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Lattice worldline representation of correlators in a background field

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Abstract: We use a discrete worldline representation in order to study the continuum limit of the one-loop expectation value of dimension two and four local operators in a background field. We illustrate this technique in the case of a scalar field coupled to a non-Abelian background gauge field. The first two coefficients of the expansion in powers of the lattice spacing can be expressed as sums over random walks on a d-dimensional cubic lattice. Using combinatorial identities for the distribution of the areas of closed random walks on a lattice, these coefficients can be turned into simple integrals. Our results are valid for an anisotropic lattice, with arbitrary lattice spacings in each direction.

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1 Introduction

1.1 Motivation

The classical statistical approximation (CSA) is an approximate scheme to study in real time the dynamics of a system of fields, as an initial value problem. It has been used in cosmology [1–3], in high energy nuclear physics to study heavy ion collisions [4–6], in studies of the Schwinger mechanism [7, 8], and in cold atom physics [9, 10]. In this approximation, the time evolution of the fields is classical (i.e. deterministic), and one averages over fluctuations of their initial conditions. Obviously, this scheme neglects all the quantum effects that would normally affect the time evolution of a system. In the path integral language, it corresponds to taking the saddle point of the integral. This can be justified if the typical field amplitude in the system under consideration is large, so that the commutator of a pair of fields is much smaller than the typical field squared.

Some quantum effects can nevertheless be included in the classical statistical approximation via the fluctuations of the initial fields. Indeed, it can be shown on general grounds that the leading (i.e. of order $\hbar$) quantum effects come entirely from the initial condition for the density operator of the system, while the quantum corrections that alter its time evolution only start at the order $\hbar^2$. In fact, there is a unique statistical ensemble of initial classical fields such that the CSA coincides with the exact $\mathcal{O}(\hbar)$ result for all observables [1, 11, 12]. These initial fields have a flat spectrum in momentum space, that extends to arbitrarily large momenta.

Because the CSA implemented in this manner[1] contains all the $\mathcal{O}(\hbar)$ contributions of the underlying theory, it also contains all their singularities, and in particular the ultraviolet divergences. In addition, it contains some of the higher order contributions, but not all of them since the quantum corrections to the time evolution are missing in this approximation.

[1] Another common implementation of the CSA is to use a spectrum of field fluctuations with a compact (or at least falling faster than $1/k$) momentum spectrum [4, 5]. This type of initial distribution corresponds to a classical ensemble of quasiparticle excitations, instead of quantum fluctuations. This version of the CSA is free of any ultraviolet divergences (for a spectrum that falls like k^{-1}, one gets some ultraviolet divergences, but the resulting approximation is super-renormalizable [13, 14]), but it also does not contain anything quantum. It may coincide with the underlying theory at tree level, but not beyond.

It has been shown recently that this leads to a dependence on the ultraviolet cutoff that cannot be disposed of by the usual renormalization of the parameters of the theory. This can be seen in computer simulations using the CSA [15], by a perturbative analysis of the graphs that arise in the CSA [16], and also from the study of cutoff effects in the classical approximation of the Boltzmann equation [17].

At the moment, it is not known whether the CSA can be modified in order to remove these unwanted terms. However, regardless of this interesting theoretical question, it is important to have a good understanding of the structure of the standard 1-loop ultraviolet divergences. Indeed, since they are identical in the CSA and in the exact theory, they can be removed by the usual renormalization procedure. But their form may be quite complicated in the lattice formulation of the CSA, especially for a generic lattice that may have anisotropic lattice spacings.[2] Generically, this requires the calculation at one-loop of the expectation value of interest, with lattice regularization and propagators, in the presence of a non-Abelian background field. Unfortunately, lattice perturbation theory is quite complicated, even for this seemingly simple task (see [18] for a review). The main issue is the treatment of the background field, and the fact that one recovers gauge invariant results by combining several pieces that are not individually gauge invariant (see the appendix E for an example of such calculation in the present context).

In the present paper, we pursue a different approach in order to obtain these 1-loop quantities, based on the so-called worldline formalism. Historically, this formalism emerged from ideas based on the limit of infinite tension in string theory[3] [20, 21], and it soon appeared that it provides a powerful way of organizing field theory calculations, especially when gauge symmetry is part of the game. A pure field theory understanding of this formalism was later proposed in ref. [22], by a method which is similar to Schwinger's proper time representation. For reviews on this approach, the reader may consult refs. [23, 24]. This formalism has been applied to the evaluation of effective actions [25–28], to the study of pair production by an external field [29, 30] or the Casimir effect [31, 32]. Numerical algorithms based on this formalism have also been proposed [32–34]. Since our goal is to apply this formalism to a lattice field theory, the closest work we are aware of is in refs. [35, 36], where a new method for evaluating functional determinants in terms of worldlines was proposed.

In the present paper, we use this formalism in order to obtain useful expressions for 1-loop expectations values in a lattice field theory coupled to a (fixed) gauge background. As we shall see, the worldline formalism is well suited for this application because it enables one to have only gauge invariant objects at all stages of the calculation. Then, we use these expressions in order to study the limit of small lattice spacing. In this limit, we obtain an expansion in terms of the background field strength, with coefficients that are given by sums over closed loops on the lattice, weighted by powers of the area enclosed by the loop. Thus, the worldline formalism relates the coefficient of the short distance expansion to combinatorial properties of closed loops on a cubic lattice.

[2] In applications to heavy ion collisions, it is common to have a much smaller lattice spacing in the direction of the collision axis.

[3] An earlier example of this approach is a string-inspired calculation of the 1-loop β function of Yang-Mills theory [19].

In the simple case where the lattice spacings are the same in all the directions, the combinatorial formulas we need were already known and can be found in ref. [37]. The generalization to anisotropic lattice spacings requires some combinatorial formulas that we could not find in the literature. A numerical exploration of all the random walks of length ≤ 20 led us to conjecture a number of such formulas (discussed in the appendix C), that extend and generalize those of ref. [37]. A proof of these formulas[4] is presented in a separate paper, ref. [38].

1.2 Model

In order to keep things rather simple and focus on the main aspects of the worldline formalism, we consider a complex scalar field coupled to an external non-Abelian gauge field. This background field is given once for all, and does not fluctuate. We neglect the self-interactions of the scalar field. The Lagrangian reads:

$$\mathcal{L} \equiv \sum_{\mu=1}^{d} (D_\mu \phi)^* (D^\mu \phi)\,, \tag{1.1}$$

where $D_\mu \equiv \partial_\mu - igA_\mu$ is the covariant derivative in the presence of the background field. We also assume that the system is initialized at $x^0 = -\infty$ into the perturbative vacuum.

We consider the expectation values of local gauge invariant operators made of the field ϕ, e.g. $\phi^*\phi$, $\phi^* D_\mu D^\mu \phi$, $(D_\mu\phi)^*(D_\nu\phi)$. With the Lagrangian given in eq. (1.1) and the vacuum state as initial condition, these expectation values are given by a 1-loop graph in a background field. These loops contain a pure vacuum contribution which is ultraviolet divergent. In addition, for operators that have a sufficiently high dimension, there may be subleading ultraviolet divergences, whose structure is however strongly constrained by gauge invariance. Our goal in this paper is to investigate the structure of these divergences, with a lattice regularization.

We work with an Euclidean metric, in d space-time dimensions, on a discrete cubic lattice. For definiteness, the spatial directions are chosen to be $1, \cdots, d-1$ and the direction d can be considered as the time direction (although this distinction is hardly relevant with an Euclidean metric). Our goal is to study the general case of arbitrary lattice spacings $a_1, \cdots, a_d$ in each direction, but in the sections 2 and 3 we expose the formalism with an isotropic lattice for simplicity. The coordinates are labeled x_1 to x_d, and we denote by $\hat{1}, \cdots, \hat{d}$ the vectors corresponding to one lattice spacing in each of the directions. Therefore, any point on the lattice can be represented as $x = x_1\hat{1} + \cdots + x_d\hat{d}$, where the x_i are integers.

1.3 Preview of the results

In d dimensions and for completely arbitrary lattice spacings in each direction, the expectation value of $|\phi(0)|^2$ at 1-loop in a background field admits the following expansion in

[4] Eqs. (C.1) and (C.5) can be viewed as "unsummed" versions of the formulas (1.5) and (1.6) of ref. [37], while eqs. (C.2), (C.3), (C.4), (C.6), (C.7) and (C.8) seem to be totally new. The appendix E provides an indirect proof of eq. (C.1), since we rederive the expansion of $\langle\phi_a^*(0)\phi_a(0)\rangle$ –that relies on eq. (C.1) in the worldline approach– using lattice perturbation theory.

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powers of the lattice spacings,

$$\langle\phi_a^*(0)\phi_a(0)\rangle \underset{\{a_i\to 0\}}{=} \frac{\mathbf{a}^2}{2d\prod_{i=1}^d a_i}\left[\mathrm{tr}_{\mathrm{adj}}(1)\,\boldsymbol{C}_0 - \frac{g^2}{4}\sum_{i<j} a_i^2 a_j^2 F_a^{ij}(0)F_a^{ij}(0)\;\boldsymbol{C}_4^{ij;ij}+\cdots\right], \tag{1.2}$$

where F_a^{ij} is the field strength associated to the background gauge field. The numerical coefficients that appear in this expansion are given by the following integrals:

$$\boldsymbol{C}_0 \equiv \int_0^\infty dt\; e^{-t}\prod_{r=1}^d I_0\Big(\frac{h_r t}{d}\Big) \tag{1.3}$$

and

$$\boldsymbol{C}_4^{ij;ij} = \frac{h_i h_j}{12 d^2}\int_0^\infty dt\; e^{-t}\;t^2\left[\prod_{k\neq i,j} I_0\Big(\frac{h_k t}{d}\Big)\right] I_1\Big(\frac{h_i t}{d}\Big)\;I_1\Big(\frac{h_j t}{d}\Big). \tag{1.4}$$

I_0 and I_1 are modified Bessel functions of the first kind. In all these equations, we denote

$$\mathbf{a}^{-2} \equiv \frac{1}{d}\sum_{i=1}^d a_i^{-2} \qquad h_i \equiv \frac{\mathbf{a}^2}{a_i^2}\,. \tag{1.5}$$

These formulas are the archetype of the results obtained in this paper. We also derive similar formulas for bilocal operators of the form $\langle\phi_a^*(0)\mathcal{W}_{ab}(\gamma_{x0})\phi_b(x)\rangle$, where the point x is separated from the origin by 1 or 2 lattice spacings. In these operators, $\mathcal{W}_{ab}(\gamma_{x0})$ is a Wilson line along a path γ_{x0} connecting x to 0, which is needed in order to have a gauge invariant operator. We shall see that the leading term of the expansion in powers of the lattice spacings does not depend on the choice of this path, while the second term in general depends on this choice.

In the rest of this paper, we use the lattice worldline formalism in order to demonstrate these formulas for the coefficients of the expansion. We first obtain intermediate representations of these coefficients in terms of sums over all the closed random walks on the lattice, which relate their values to some combinatorial properties of random walks. These formulas can then be transformed into the integral representations listed above, by using the 2-dimensional combinatorial formulas of the appendix C.

1.4 Outline of the paper

In the section 2, we derive in detail the worldline formulation of the expectation value $\langle\phi_a^*(0)\phi_a(0)\rangle$ on a lattice with isotropic spacings, and its short distance expansion. A subsection is devoted to the discussion of infrared divergences and their manifestation in the worldline formalism. We also introduce in this section the Borel transformation that turns the combinatorial sums into integrals. In the section 3, we extend this study to bilocal operators, i.e. operators that contain a ϕ^* and a ϕ evaluated at separate lattice spacings. This extension is of great practical importance, because these operators appear in the discretization of covariant derivatives. The section 4 generalizes all the previous results to a more general lattice setup, where each direction of space-time has its own lattice spacing. As an illustration, we study the limit where one of the lattice spacings is much smaller than the others, and we apply this to a discussion of the energy-momentum tensor. The section 5 is devoted to concluding remarks.

A number of more technical aspects are discussed in several appendices. In the appendix A, we show how this formalism is modified on a finite lattice with periodic boundary conditions (in the main part of the paper, we take the limit of zero lattice spacing at fixed physical volume, so that the size of the lattice becomes infinite and the boundary conditions are irrelevant). In the appendix B, we derive the leading coefficient of the expansion for bilocal operators with an arbitrary separation between the two fields. The appendix C discusses all the combinatorial formulas that are necessary in the case of anisotropic lattices, and in D we recall the connection between the statistics of the areas of closed loops on a two-dimensional lattice and the spectral properties of the so-called almost-Mathieu operator. In the appendix E, we obtain the short distance expansion of $\langle\phi_a^*(0)\phi_a(0)\rangle$ from lattice perturbation theory, mainly to illustrate the technical complexity of this approach. In the appendix F, we study from the outset a hybrid description in which one of the directions (e.g. time) is treated as a continuous variable, while the others remain discretized and we show that this is equivalent to starting from a fully discrete description and taking one lattice spacing to zero.

2 Local operator $\langle\phi_a^*(0)\phi_a(0)\rangle$

2.1 Discrete heat kernel

In order to establish the formalism, consider first the expectation value of the operator $\langle\phi_a^*(0)\phi_a(0)\rangle$ (the color indices are summed to ensure gauge invariance) at one loop. To lighten the notations, we evaluate the expectation value at the point $x^\mu = 0$, but all our considerations are completely general. Since this expectation value is given by a 1-loop graph, we can first write

$$\langle\phi_a^*(0)\phi_a(0)\rangle = \langle x^\mu = 0\big|\frac{1}{D^2}\big|x^\mu = 0\rangle\,. \tag{2.1}$$

The standard heat kernel approach would be to write

$$\frac{1}{D^2} = \int_0^\infty ds\;\exp(-s\,D^2)\,. \tag{2.2}$$

However, in our case it is more convenient to use a discrete version of this formula,[5]

$$\frac{2d}{a^2D^2} = \sum_{n=0}^\infty (1 - a^2D^2/2d)^n\,, \tag{2.3}$$

which is also exact. We have multiplied D^2 by the lattice spacing squared in order to get a dimensionless combination. The purpose of the factor $2d$ (where d is the number of space-time dimensions) will become clearer later on.

[5] Up to a rescaling, the integer n is a discrete version of the integration variable s.

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Consider now a sequence of functions $P_n(x)$ defined on the lattice, and satisfying the following iteration rule:

$$P_{n+1} = (1 - a^2 D^2/2d)\, P_n\,. \tag{2.4}$$

If we define $P_0(x) = \delta_x$, then we have

$$\langle 0 \big| (1 - a^2 D^2/2d)^n \big| 0 \rangle = a^{-d}\, P_n(x)\,. \tag{2.5}$$

If we interpret $P_0(x)$ as a probability distribution localized at the point $x^\mu = 0$, then P_n is the probability distribution after n iterations of the process described in eq. (2.4). In other words, it is the probability that this process starts and returns at the point $x^\mu = 0$ after exactly n steps.

2.2 Vacuum case

Eq. (2.4) may be rewritten as

$$P_{n+1} - P_n = -\frac{a^2}{2d} D^2 P_n\,. \tag{2.6}$$

If we view the index n as a discrete fictitious time, and if the metric is Euclidean, then this is a discrete diffusion equation and the evolution of the probability distribution P_n can be remapped in terms of random walks.

For illustration purposes, consider first the free case. We have

$$-\frac{a^2}{2d} D^2 f(i, \cdots) = -f(i, \cdots) + \frac{f(i+1, \cdots) + f(i-1, \cdots)}{2d} + \cdots\,. \tag{2.7}$$

The eq. (2.6) can then be written more explicitly as

$$P_{n+1}(i, \cdots) = \frac{P_n(i+1, \cdots) + P_n(i-1, \cdots)}{2d} + \cdots\,, \tag{2.8}$$

where the sum in the numerator extends to all the nearest neighbors. This equation describes a random walk where at each step one moves to one of the adjacent sites of the lattice with probability $1/2d$. We see now the reason for the peculiar normalization[6] in eq. (2.1): by doing this, we can eliminate the term proportional to $P_n(i, \cdots)$ in the right hand side, i.e. the possibility for the random walk process to stall during the step.[7]

$P_n(0)$ is the probability that such a random walk returns at the point 0 after n steps. Geometrically, this means that the random walk is a closed loop of length n. Since at each step, there are two possibilities to move in each direction, the total number of random walks of length n is $(2d)^n$. $P_n(x)$ is thus the number of *closed* random walks of length n, divided by the total number $(2d)^n$. Therefore, we can write

$$\langle \phi_a^*(0)\phi_a(0) \rangle = \frac{1}{2da^{d-2}} \sum_{n=0}^{\infty} \frac{1}{(2d)^{2n}} \sum_{\gamma \in \Gamma_{2n}(0,0)} \mathrm{tr}_{\mathrm{adj}}\,(1)\,, \tag{2.9}$$

[6] Alternatively, one could view this normalization as choosing a specific ratio between the size of the steps in the fictitious time and the lattice spacing.

[7] By excluding the possibility that the random walk stalls, we ensure that the number of steps n is also the length of the path.

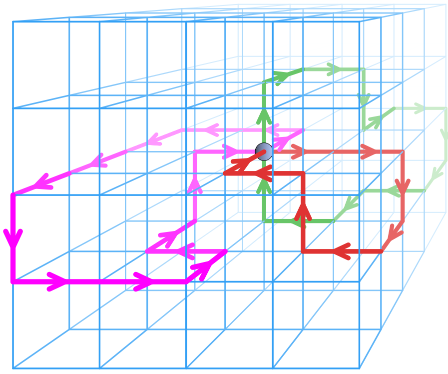

Figure 1. Example of closed random paths on a 3-dimensional cubic lattice. The blob indicates the location of the base point 0.

where $\Gamma_{2n}(0,0)$ is the set of all closed random walks of base point 0 (i.e. starting and ending at 0) and length $2n$ on the lattice (the length of such a closed path must be an even number). A few of the closed paths involved in eq. (2.9) are illustrated in the figure 1.

In the vacuum, the double sum is independent of the lattice spacing. It is just a pure number that sets the normalization of the result. The trace in the adjoint representation comes from the summation over the color indices, and brings a factor $N_c^2 - 1$.

2.3 Non-zero background field

On the lattice, the background field is represented in terms of compact link variables $U_i(x)$ in order to preserve an exact gauge invariance despite the discretization. In terms of these link variables, the covariant derivative squared becomes,

$$-\frac{a^2}{2d}D^2 f(i,\cdots) = -f(i,\cdots) + \frac{U_1(i,\cdots)f(i+1,\cdots) + U_1^{-1}(i-1,\cdots)f(i-1,\cdots)}{2d} + \cdots . \tag{2.10}$$

Therefore, when the links are not unity, the random walk is biased by the background field. The end result is that eq. (2.9) is modified into

$$\langle \phi_a^*(0)\phi_a(0)\rangle = \frac{1}{2da^{d-2}} \sum_{n=0}^{\infty} \frac{1}{(2d)^{2n}} \sum_{\gamma\in\Gamma_{2n}(0,0)} \mathrm{tr}_{\mathrm{adj}}\left(\mathcal{W}(\gamma)\right). \tag{2.11}$$

In words, the $\mathrm{SU}(N_c)$ identity matrix in eq. (2.9) is replaced by a Wilson loop $\mathcal{W}(\gamma)$ obtained by multiplying all the link variables along the closed contour γ. This formula is manifestly gauge invariant, since Wilson loops are gauge invariant. Note also that this formula is exact at 1-loop on the lattice.

2.4 Notations and basic facts about closed random walks

In the previous subsections, we have introduced $\Gamma_{2n}(0,0)$, the set of all the paths of length $2n$ drawn on the lattice, with endpoints 0 and 0 (i.e. closed paths). More generally, we will denote $\Gamma_n(0,x)$ the set of paths of length n from 0 to a point x (all these paths have

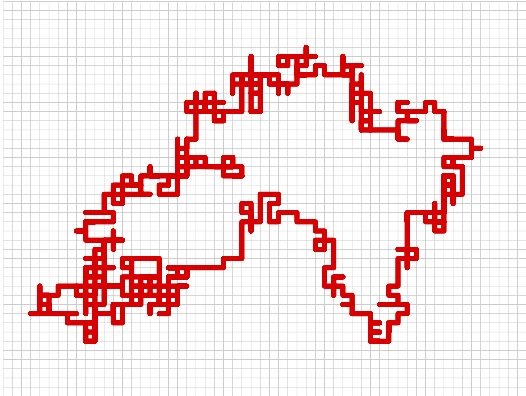

Figure 2. Closed random walk on a square lattice in two dimensions. The length of the path is $2na$, while the diameter of the domain explored by the random walk is of order $\sqrt{n}a$ and its area is of order na^2.

the same parity, which is also the parity of the sum of the coordinates of the point x). To avoid encumbering the notation, we do not specify the dimension of the lattice in which these paths should be considered, since the context of the formula in which the notation appears is sufficient to make this obvious.

When we need an explicit representation for a path, we denote it by the sequence of the hops it contains, such as

$$\gamma = \hat{2}\,\hat{4}\,\hat{3}^{-1}\cdots, \tag{2.12}$$

(read from left to right.) The notation $\hat{3}^{-1}$ denotes a hop in the $-x_3$ direction. The empty path will be denoted $\gamma = \mathbf{1}$, and the concatenation of two paths γ_1 and γ_2 is denoted by $\gamma = \gamma_1 \otimes \gamma_2$ (read again from left to right, so that γ_1 is the first part of the resulting path). Obviously, $\mathbf{1} \otimes \gamma = \gamma \otimes \mathbf{1} = \gamma$.

In two dimensions, we will also introduce later in the paper the subset $\mathbf{\Gamma}_{n_1,n_2}(0,x)$, made of all the paths connecting 0 to x and making exactly n_1 hops in the $+x_1$ direction and n_2 hops in the $+x_2$ direction. The numbers $n'_{1,2}$ of hops in the opposite directions, $-x_1$ and $-x_2$, do not need to be specified explicitly since it can be inferred from $n_{1,2}$ and the coordinates of the point x. Indeed, if $x = x_1\hat{1} + x_2\hat{2}$, we have

$$n'_1 = n_1 - x_1\,, \qquad n'_2 = n_2 - x_2\,. \tag{2.13}$$

Obviously, $\Gamma_n(0,x)$ and $\mathbf{\Gamma}_{n_1,n_2}(0,x)$ are related by

$$\Gamma_n(0,x) = \bigcup_{n_1+n_2=\frac{n+x_1+x_2}{2}} \mathbf{\Gamma}_{n_1,n_2}(0,x)\,. \tag{2.14}$$

In order to develop some intuition with formulas such as eq. (2.11), let us recall here some elementary properties of closed random walks. Let us consider a closed random walk made of $2n$ hops, all illustrated in the figure 2 in two dimensions. For such a random walk, one has the following properties:

i. the length of the path is obviously $2na$,

ii. the typical size of the domain explored by the random walk grows only like $\sqrt{n}a$,

iii. the area enclosed by the random walk (or the area of its projection on a plane in $d > 2$ dimensions) grows as na^2.

The property **ii** plays a role in the infrared behavior of the quantity under consideration. Indeed, as we shall see later, infrared singularities arise when the contribution of "large" random walks does not decrease fast enough. Similarly, **iii** plays a role in the second term in the expansion in powers of the lattice spacing.

2.5 Continuum limit $a \to 0$

If we let the lattice spacing a go to zero, while the background is held fixed in physical units, the Wilson loop $\mathcal{W}(\gamma)$ goes to the identity because the closed loop γ shrinks to a tiny loop of base point 0. Therefore, it can be approximated by the exponential of the magnetic flux across a surface Σ whose boundary is γ (this surface is a tiling of elementary lattice squares),

$$\mathcal{W}(\gamma) \underset{a\to 0}{\approx} \exp\left\{ iga^2 \sum_{\mu<\nu} A_{\mu\nu}(\gamma) F_a^{\mu\nu}(0) t^a \right\}, \tag{2.15}$$

where $A_{\mu\nu}(\gamma)$ is the algebraic area, measured as a number of plaquettes since we have already pulled out a factor a^2, of the domain enclosed by the projection of the contour γ on the (μ, ν) plane. The orientation of γ dictates the orientation of the projection, which in turn controls the sign of $A_{\mu\nu}(\gamma)$. Note that in this limit, the field strength can be considered uniform across the entire lattice, and therefore $F^{\mu\nu}$ is evaluated at the point $x^\mu = 0$.

Since the loop size tends to zero when $a \to 0$, we can do a Taylor expansion of the exponential. In order to get a non-trivial answer after taking the trace, we must go to second order:

$$\mathrm{tr}_{\mathrm{adj}}\left(\mathcal{W}(\gamma)\right) \approx \mathrm{tr}_{\mathrm{adj}}\left(1\right) - \frac{g^2 a^4}{4} \left\{ \sum_{\mu<\nu} A_{\mu\nu}(\gamma) F_a^{\mu\nu}(0) \right\} \left\{ \sum_{\rho<\sigma} A_{\rho\sigma}(\gamma) F_a^{\rho\sigma}(0) \right\} \tag{2.16}$$

(A factor 1/2 comes from $\mathrm{tr}_{\mathrm{adj}}(t^a t^b) = \frac{1}{2}\delta^{ab}$.) By plugging this in the formula (2.11), we obtain the following expansion

$$\langle \phi_a^*(0)\phi_a(0)\rangle \approx \frac{1}{2d\, a^{d-2}} \left[\boldsymbol{C}_0\, \mathrm{tr}_{\mathrm{adj}}(1) - \frac{g^2 a^4}{4} \sum_{\substack{\mu<\nu \\ \rho<\sigma}} F_a^{\mu\nu}(0) F_a^{\rho\sigma}(0)\, \boldsymbol{C}_4^{\mu\nu;\rho\sigma} \right] \tag{2.17}$$

where the coefficients $\boldsymbol{C}_0$ and $\boldsymbol{C}_4^{\mu\nu;\rho\sigma}$ are purely geometrical quantities defined by sums over all the closed loops on the lattice

$$\begin{aligned} \boldsymbol{C}_0 &\equiv \sum_{n=0}^{\infty} \frac{1}{(2d)^{2n}} \sum_{\gamma\in\Gamma_{2n}(0,0)} 1 \\ \boldsymbol{C}_4^{\mu\nu;\rho\sigma} &\equiv \sum_{n=0}^{\infty} \frac{1}{(2d)^{2n}} \sum_{\gamma\in\Gamma_{2n}(0,0)} A_{\mu\nu}(\gamma) A_{\rho\sigma}(\gamma)\,. \end{aligned} \tag{2.18}$$

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(Note that in these formulas, we have replaced $n \to 2n$ since only random paths of even length can be closed.) The equation (2.17) provides an explicitly gauge invariant expansion in powers of the lattice spacing. The coefficients (that remain to be calculated) are geometrical quantities that depend on the dimension and the lattice under consideration, but not on the background field.

In the second of eqs. (2.18), $A_{\mu\nu}(\gamma)$ is the area of the surface enclosed by γ projected on the $\mu\nu$ plane. Several remarks are in order about this quantity:

- These areas are "algebraic", in the sense that they may have a sign that takes into account the orientation of the boundary, and a multiplicity that depends on the winding number.
- There are many surfaces with the same boundary γ. $A_{\mu\nu}(\gamma)$ does not depend on this choice but only on the boundary.
- They do not depend on the base point $x^\mu = 0$. Specifying a base point is only necessary in order to avoid counting multiple times loops that have the same shape up to a translation.

From eq. (2.17) one can read the structure of ultraviolet divergences in the operator $\phi^*\phi$. The leading ultraviolet divergence is a power divergence in a^{2-d}. In a massless theory, the next term is of order a^{6-d} and is thus ultraviolet finite in $d < 6$ dimensions. Indeed, the gauge invariance and locality of this operator imposes that the expansion involves only gauge invariant local combinations of the background field. The first non-trivial such quantity is the square of the field strength, that has dimension 4. Therefore, the second term differs from the leading term by a factor a^4. As we shall see in the section 2.8, the expectation value of $\phi^*\phi$ in a massless theory also contains infrared divergences, that would be regularized by the introduction of a mass for the scalar field.

2.6 Zeroth order coefficient

It is possible to provide an integral expression for the coefficient $\boldsymbol{C}_0$, starting from the combinatorial formula that explicitly counts the number of closed random walks in terms of the number of hops in the d directions (respectively $2n_1, 2n_2, \cdots, 2n_d$),

$$\boldsymbol{C}_0 = \sum_{n=0}^{\infty} \frac{(2n)!}{(2d)^{2n}} \sum_{n_1+\cdots+n_d=n} \frac{1}{n_1!^2 \cdots n_d!^2}\,. \tag{2.19}$$

The factor $(2n)!$ prevents the complete separation of the sums over the n_i. However, it can be removed by a Borel transformation:

$$\boldsymbol{C}_0 = \int_0^\infty dt\, e^{-t}\, A_d\Big(\frac{t}{2d}\Big)\,, \tag{2.20}$$

where we denote

$$A_d(x) \equiv \sum_{n=0}^{\infty} x^{2n} \sum_{n_1+\cdots+n_d=n} \frac{1}{n_1!^2 \cdots n_d!^2} = \Bigg[\sum_{p=0}^{\infty} \frac{x^{2p}}{p!^2}\Bigg]^d = I_0^d(2x)\,, \tag{2.21}$$

where I_0 is a modified Bessel function of the first kind. Therefore, we have

$$\boldsymbol{C}_0 = \int_0^\infty dt\, e^{-t}\, I_0^d\left(\frac{t}{d}\right). \tag{2.22}$$

In 3 dimensions, this leads to an explicit formula ([39] — section 6.612.6):

$$\boldsymbol{C}_0 \underset{d=3}{=} \frac{\sqrt{3}-1}{32\pi^3}\,\Gamma^2\left(\frac{1}{24}\right)\Gamma^2\left(\frac{11}{24}\right) \approx 1.51638606\,, \tag{2.23}$$

while in 4 dimensions we have only been able to evaluate it numerically,

$$\boldsymbol{C}_0 \underset{d=4}{\approx} 1.23946712\,. \tag{2.24}$$

2.7 Variance of the areas of closed random walks

In the second term of the expansion in powers of the lattice spacing, we need the quantity

$$\boldsymbol{C}_4^{\mu\nu;\rho\sigma} \equiv \sum_{n=0}^{\infty} \frac{1}{(2d)^{2n}} \sum_{\gamma\in\Gamma_{2n}(0,0)} A_{\mu\nu}(\gamma)A_{\rho\sigma}(\gamma)\,. \tag{2.25}$$

A central result for the rest of our discussion in the isotropic case is the value of this sum in two dimensions. In $d = 2$, the variance of the algebraic areas enclosed by closed random walks of length $2n$ is given by the following formula [37] (eqs. (1.4)–(1.5)):

$$\left[\sum_{\gamma\in\Gamma_{2n}(0,0)} (A_{12}(\gamma))^2\right]_{\dim\ 2} = \binom{2n}{n}^2 \frac{n^2(n-1)}{6(2n-1)}\,. \tag{2.26}$$

This formula is all we need in order to evaluate the coefficient $\boldsymbol{C}_4$ for $d = 2$. But note that this coefficient diverges in $d = 2$: using Stirling's asymptotic formula for the factorial, one can see that the sum over the length $2n$ of the path is divergent (in this formalism, this is the counterpart of an infrared divergent loop integral in low dimension).

In higher dimensions, the first thing to notice is that it is sufficient to consider $\mu = \rho, \nu = \sigma$ (if there is a mismatch of the indices, the average over all closed loops gives zero since the area is signed). For the sake of definiteness, let us choose $\mu = 1, \nu = 2$. For a given closed loop γ, the area $A_{12}(\gamma)$ is the area of its projection on the 12 plane. Every closed random walk in d dimensions can be decomposed into hops that are in the 12 plane, and hops orthogonal to this plane (see the figure 3). The latter disappear in the projection on the 12 plane, and therefore do not play any role in the calculation of the area $A_{12}(\gamma)$. Moreover, the projection of γ in the 12 plane is itself a closed loop in 2 dimensions, while the sequence of the transverse hops is a closed loop in $d-2$ dimensions. Let us denote $2n$ the number of hops in the 12 plane and $2m$ the number of hops in the transverse directions. One can rewrite the coefficient $\boldsymbol{C}_4$ as follows

$$\boldsymbol{C}_4^{12;12} \equiv \sum_{m,n=0}^{\infty} \frac{1}{(2d)^{2(n+m)}} \binom{2(m+n)}{2m} \times \left[\sum_{\sigma\in\Gamma_{2m}(0,0)} 1\right]_{\dim\ d-2} \times \left[\sum_{\gamma\in\Gamma_{2n}(0,0)} (A_{12}(\gamma))^2\right]_{\dim\ 2}. \tag{2.27}$$

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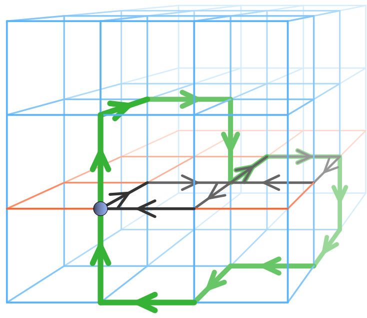

Figure 3. Example of closed random walk on a 3-dimensional cubic lattice, with $2n = 8$ hops in the 12 plane and $2m = 4$ hops in the 3-rd direction. The 12 plane is highlighted in orange, and the projection of the closed loop on the plane is shown in gray (it has an area $A_{12} = -2$ in this example). The blob indicates the location of the base point 0.

The binomial factor in the first line counts the number of ways to intertwine the $2m$ transverse hops and the $2n$ in-plane hops. In the second line, the first factor is the number of length $2m$ closed random walks in $d-2$ dimensions, and the second factor is the squared area summed over all closed loops of length $2n$ in two dimensions. This latter factor is given by eq. (2.26). For the first factor, since we are interested primarily in $d = 3$ $(d-2 = 1)$ and $d = 4$ $(d-2 = 2)$, we can use the following standard results

$$\begin{aligned}\Big[\sum_{\sigma\in\Gamma_{2m}(0,0)} 1\Big]_{\text{dim }1} &= \binom{2m}{m}\\ \Big[\sum_{\sigma\in\Gamma_{2m}(0,0)} 1\Big]_{\text{dim }2} &= \binom{2m}{m}^2 .\end{aligned} \tag{2.28}$$

Therefore, for these dimensions, we find the following expressions for the coefficient $\boldsymbol{C}_4^{12;12}$,

$$\boldsymbol{C}_4^{12;12} \underset{d=3}{=} \sum_{l=0}^{\infty} \frac{(2l)!}{6^{2l}} \sum_{n=0}^{l} \frac{(2n)!}{(l-n)!^2 n!^4} \frac{n^2(n-1)}{6(2n-1)} , \tag{2.29}$$

and

$$\boldsymbol{C}_4^{12;12} \underset{d=4}{=} \sum_{l=0}^{\infty} \frac{(2l)!}{8^{2l}} \sum_{n=0}^{l} \frac{(2(l-n))!(2n)!}{(l-n)!^4 n!^4} \frac{n^2(n-1)}{6(2n-1)} . \tag{2.30}$$

For a given total length $2l = 2m + 2n$ of the random walks, the variance of the projected algebraic area is thus expressed as a sum of $l+1$ terms, whose evaluation is very easy (especially compared to a direct evaluation by exhausting the list of all random walks of this length, since there are $(2d)^{2l}$ such walks). In the table 1, we list as an example the summands (at fixed l) for the 3 dimensional case (eq. (2.29)). We have performed an exhaustive search of all the closed random walks up to $2l = 14$, and we have checked the agreement between this direct computation and the formula (2.29). The values listed for $2l > 14$ were solely obtained from eq. (2.29).

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$2l$	#(paths)	#(closed paths)	$\frac{\sum_{\gamma\in\Gamma_{2l}(0,0)}(A_{12}(\gamma))^2}{6^{2l}}$
2	36	6	0.0000000000
4	1296	90	0.0061728395
6	46656	1860	0.0102880658
8	1679616	44730	0.0133363816
10	60466176	1172556	0.0158369532
12	2176782336	32496156	0.0180064306
14	78364164096	936369720	0.0199490385
20	36^{10}		0.0249038527
100	36^{50}		0.0600254031
200	36^{100}		0.0856471034
1000	36^{500}		0.1928668060
2000	36^{1000}		0.2729940025

Table 1. Exhaustive enumeration of random walks and closed random walks in 3 dimensions, up to the length $2l = 14$. The last column gives the corresponding contribution to $\boldsymbol{C}_4^{12;12}$. The values for $2l > 14$ are obtained from eq. (2.29).

It is also possible to establish an integral representation of $\boldsymbol{C}_4^{12;12}$, valid in any dimension d, similar to eq. (2.22) for $\boldsymbol{C}_0$. The first step is to introduce the combinatorial representation for the factor that counts the closed random walks of length $2m$ in $d-2$ dimensions. This leads to

$$\boldsymbol{C}_4^{12;12} = \sum_{m,n=0}^{\infty} \frac{1}{(2d)^{2(m+n)}} \frac{(2(m+n))!}{(2m)!(2n)!} \sum_{m_1+\cdots+m_{d-2}=m} \frac{(2m)!}{m_1!^2\cdots m_{d-2}!^2} \times \frac{(2n)!^2}{n!^4} \frac{n^2(n-1)}{6(2n-1)} \,. \tag{2.31}$$

We can separate the sums over m and n by a Borel transformation,

$$\boldsymbol{C}_4^{12;12} = \int_0^\infty dt\; e^{-t}\; C_d\left(\frac{t}{2d}\right), \tag{2.32}$$

with

$$\begin{aligned} C_d(x) &\equiv \sum_{m,n=0}^{\infty} \frac{x^{2(m+n)}}{(2m)!(2n)!} \sum_{m_1+\cdots+m_{d-2}=m} \frac{(2m)!}{m_1!^2\cdots m_{d-2}!^2} \frac{(2n)!^2}{n!^4} \frac{n^2(n-1)}{6(2n-1)} \\ &= \frac{x^2}{3}\; I_0^{d-2}(2x)\; I_1^2(2x)\,. \end{aligned} \tag{2.33}$$

Therefore, we have

$$\boldsymbol{C}_4^{12;12} = \frac{1}{12d^2}\int_0^\infty dt\; e^{-t}\; t^2\; I_0^{d-2}\left(\frac{t}{d}\right)\; I_1^2\left(\frac{t}{d}\right). \tag{2.34}$$

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2.8 Infrared divergences

As one can see in the table 1, the summands do not decrease at large path lengths l, and the sum over l is divergent. In $d = 3$, the summand grows as $l^{1/2}$ for large path lengths, and it goes to a constant in $d = 4$. If we cut off the sum over l at some $l_{\rm max}$, this implies that

$$\boldsymbol{C}_4^{12;12} \underset{d=3}{\sim} l_{\rm max}^{3/2}\,, \qquad\qquad \boldsymbol{C}_4^{12;12} \underset{d=4}{\sim} l_{\rm max}\,. \tag{2.35}$$

This divergence for random walks that explore large regions in spacetime is the manifestation in the worldline formalism of an infrared singularity. This is corroborated by the fact the divergence is milder in $d = 4$ compared to $d = 3$. In the integral representation (2.34), this singularity appears as a divergence of the integral at large t: using the fact that $I_n(t) \sim t^{-1/2}e^t$ at large t, we see that the exponential factors cancel and that the remaining algebraic factors decrease fast enough for convergence only if $d > 6$.

In order to further investigate this, let us add a mass term[8] to the Lagrangian of the scalar field,

$$\mathcal{L} \equiv \sum_{\mu=1}^{d} (D_\mu\phi)^*(D_\mu\phi) - m^2\phi^*\phi\,. \tag{2.36}$$

In order to arrive again at a sum of random walks without stalls, one should start from the following formula

$$\frac{2\tilde{d}}{a^2(D^2+m^2)} = \sum_{n=0}^{\infty}\left(1 - a^2\frac{D^2+m^2}{2\tilde{d}}\right)^n, \tag{2.37}$$

where we have defined $\tilde{d} \equiv d + \frac{1}{2}m^2a^2$. Most of the discussion is unchanged and eq. (2.11) becomes

$$\langle\phi_a^*(0)\phi_a(0)\rangle = \frac{1}{2\tilde{d}a^{d-2}}\sum_{n=0}^{\infty}\frac{1}{(2\tilde{d})^n}\sum_{\gamma\in\Gamma_n(0,0)} {\rm tr}_{\rm adj}\left(\mathcal{W}(\gamma)\right). \tag{2.38}$$

A crucial difference is that each hop in the random walk is now weighted by a factor $1/2\tilde{d}$ instead of $1/2d$. Since $\tilde{d} > d$, this leads to an exponential reduction of the contribution of long random walks. The eqs. (2.29) and (2.30) are modified into

$$\boldsymbol{C}_4^{12;12} \underset{d=3}{=} \sum_{l=0}^{\infty}\frac{(2l)!}{(6+m^2a^2)^{2l}}\sum_{n=0}^{l}\frac{(2n)!}{(l-n)!^2n!^4}\,\frac{n^2(n-1)}{6(2n-1)}\,, \tag{2.39}$$

and

$$\boldsymbol{C}_4^{12;12} \underset{d=4}{=} \sum_{l=0}^{\infty}\frac{(2l)!}{(8+m^2a^2)^{2l}}\sum_{n=0}^{l}\frac{(2(l-n))!(2n)!}{(l-n)!^4n!^4}\,\frac{n^2(n-1)}{6(2n-1)}\,. \tag{2.40}$$

[8]One may view this mass as a temporary regulator for the infrared sector. Note that if we do not expand the observable in powers of the background field, then the background field itself would provide a natural infrared cutoff. The sum over l would naturally be cutoff when random walks reach a size comparable to the coherence length of the background field. For instance, if the background field is incoherent beyond the length scale Q^{-1}, then values $l \gtrsim (Qa)^{-2}$ are suppressed.

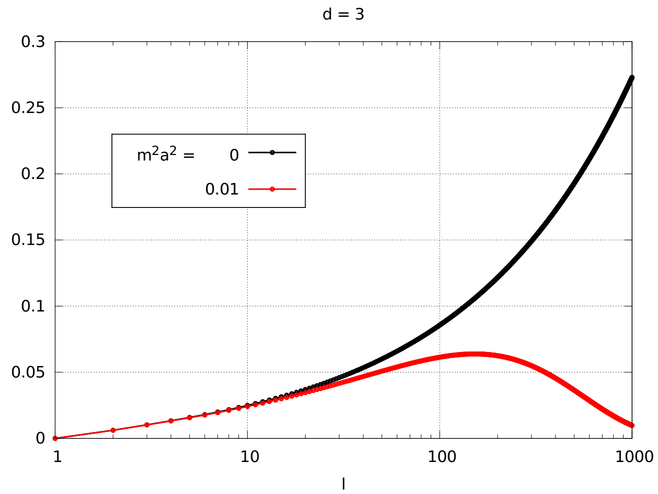

Figure 4. Summand in 3 dimensions, in the massless case (black curve) and for $m^2a^2 = 0.01$ (red curve).

These discrete sums also have an integral representation in terms of modified Bessel functions,

$$\boldsymbol{C}_4^{12;12} = \frac{1}{12\tilde{d}^2} \int_0^\infty dt\, e^{-t}\, t^2\, I_0^{d-2}\left(\frac{t}{\tilde{d}}\right) I_1^2\left(\frac{t}{\tilde{d}}\right). \tag{2.41}$$

The derivation of this formula follows the same line as that of eq. (2.34).

The modification of the behavior for large random walks is readily seen,[9]

$$\frac{1}{(2\tilde{d})^{2l}} = \frac{1}{(2d)^{2l}} \frac{1}{(1+\frac{m^2a^2}{2d})^{2l}} \approx \frac{1}{(2d)^{2l}}\, e^{-lm^2a^2/d}. \tag{2.42}$$

The regularizing effect of the mass becomes effective for lengths $l \gtrsim d/m^2a^2$. This corresponds to random walks that explore a domain of size $r \gtrsim \sqrt{d}/m$, i.e. of the order of the Compton wavelength. The values of the summand in $d = 3$ are shown for $m^2a^2 = 0.01$ in the figure 4, and compared to the massless case. With a non-zero mass, the summand increases until it reaches a maximum and then decreases exponentially, which ensures the convergence of the sum over l. Note that in doing this, we do not expand the a dependence that comes from $2d + m^2a^2$.

After this infrared regularization, the singular behaviors of eq. (2.35) would be replaced by

$$\boldsymbol{C}_4^{12;12} \underset{d=3}{\sim} (ma)^{-3}, \qquad \boldsymbol{C}_4^{12;12} \underset{d=4}{\sim} (ma)^{-2}. \tag{2.43}$$

These formulas contain inverse powers of the lattice spacing. This tells us that the expansion of eq. (2.17), in which the second term is suppressed by a^4, is upset by infrared singularities. In fact, the second term does not vanish when $a \to 0$, but instead is a term

[9] If we consider short loops instead of long ones, we can further expand the exponential in powers of lm^2a^2 and we see that the introduction of a mass generates new terms in the short distance behavior, that differ from the leading a^{2-d} term by a factor m^2a^2.

of order a^0, both in $d = 3$ and $d = 4$,

$$\langle \phi_a^*(0)\phi_a(0)\rangle \underset{d=3,4}{=} a^{2-d} \oplus m^{2-d} a^0 \,. \tag{2.44}$$

The same is true when the infrared regularization is provided by the background field rather than by a mass. In this case, the mass is replaced by the coherence scale Q of the background field in the above counting. Note that a proper calculation of the terms in a^0 requires a non-perturbative treatment of the background field (as we have seen before, it was the expansion in powers of the background field that caused the infrared divergence in the first place).

3 Bilocal operators $\langle \phi_a^*(0)\mathcal{W}_{ab}(\gamma_{x0})\phi_b(x)\rangle$

3.1 Worldline representation

Up to now, we have considered only the local operator $\phi^*(0)\phi(0)$. However, since derivatives are represented on the lattice as finite differences, we need also to consider composite operators made of two elementary fields evaluated at different lattice sites. Let us illustrate this by the discretization of the operator $\phi^* D_\mu D_\mu \phi$. On the lattice, this can be represented as follows,

$$\phi_a^*(0)(D_\mu D_\mu \phi)_a(0) = \frac{1}{a^2}\Big[\phi_a^*(0)U_\mu^{ab}(0)\phi_b(\widehat{\mu}) + \phi_a^*(0)U_\mu^{\dagger ab}(-\widehat{\mu})\phi_b(-\widehat{\mu}) - 2\phi_a^*(0)\phi_a(0)\Big] \,. \tag{3.1}$$

Here, one of the derivatives has been discretized as a forward derivative and the other as a backward one. We have used covariant derivatives for the operator to be gauge invariant. After discretization, this leads to link variables connecting the two lattice sites where the scalar field and its complex conjugate are evaluated.

All the terms in the right hand side of eq. (3.1) belong to a class of operators that contain two fields $\phi^* \cdots \phi$ linked by a Wilson line,

$$\langle \phi_a^*(0)\mathcal{W}_{ab}(\gamma_{x0})\phi_b(x)\rangle \,. \tag{3.2}$$

In this equation, γ_{x0} is a path (drawn on the edges of the lattice) connecting the points x and 0 where the two fields are evaluated. The Wilson line and the summation over the color indices ensure that this operator is gauge invariant. However, the choice of this path is arbitrary and is therefore a part of the definition of the operator under consideration. Except when the background field is a pure gauge, different paths correspond to operators that have distinct expectation values.

It is quite straightforward to generalize the derivation done in the section 2 in order to obtain the lattice worldline representation for this type of operator,

$$\langle \phi_a^*(0)\mathcal{W}_{ab}(\gamma_{x0})\phi_b(x)\rangle = \frac{1}{2da^{d-2}} \sum_{n=0}^{\infty} \frac{1}{(2d)^n} \sum_{\gamma \in \Gamma_n(0,x)} \mathrm{tr}_{\mathrm{adj}}\left(\mathcal{W}(\gamma_{x0} \otimes \gamma)\right) . \tag{3.3}$$

In this formula, $\Gamma_n(0,x)$ denotes the set of all the random walks of length n that start at the point 0 and end at the point x, and $\gamma_{x0} \otimes \gamma$ denotes the closed path obtained by concatenating one of these paths and the contour γ_{x0} used in the definition of the operator, as illustrated in the figure 5.

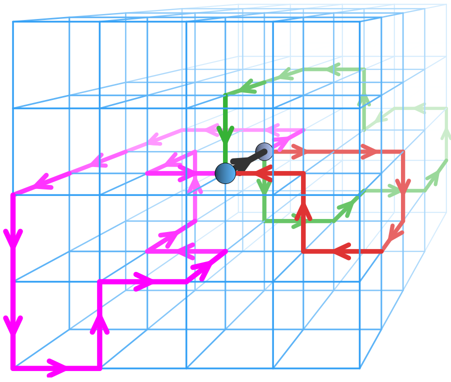

Figure 5. Example of random paths that appear in the worldline representation of bilocal operators, on a 3-dimensional cubic lattice. The blobs indicate the locations of the base points 0 and x (here separated by 1 lattice spacing). The black link is the Wilson line $\mathcal{W}(\gamma_{x0})$ inserted between the two fields for gauge invariance.

3.2 Continuum limit

Let us now consider the continuum limit of these expectation values. There are two non-equivalent ways to view this limit:

i. Keep the separation x–0 between the two points fixed in absolute units. When $a \to 0$, this interval becomes infinite in lattice units. This leads to milder ultraviolet divergences, since when the $a \to 0$ limit is performed in this way, one is in fact considering a non-local operator.

ii. Keep x and 0 at fixed locations on the lattice while $a \to 0$. Therefore, the spacing x–0 goes to zero in absolute units, and the limit $a \to 0$ corresponds to a local composite operator.

The limit **ii** is the one we consider here, since we are interested in operators such as the one in eq. (3.1). The eq. (2.17) becomes

$$\langle \phi_a^*(0)\mathcal{W}_{ab}(\gamma_{x0})\phi_b(x)\rangle = \frac{1}{2da^{d-2}}\left[\boldsymbol{D}_{0,\gamma_{x0}}\,\mathrm{tr}_{\mathrm{adj}}\,(1) - \frac{g^2a^4}{4}\sum_{\substack{\mu<\nu\\ \rho<\sigma}} F_a^{\mu\nu}(x)F_a^{\rho\sigma}(x)\;\boldsymbol{D}_{4,\gamma_{x0}}^{\mu\nu;\rho\sigma} + \cdots\right], \tag{3.4}$$

where the coefficients $\boldsymbol{D}_{0,\gamma_{x0}}$ and $\boldsymbol{D}_{4,\gamma_{x0}}^{\mu\nu;\rho\sigma}$ are generalizations of the coefficients $\boldsymbol{C}_0$ and $\boldsymbol{C}_4^{\mu\nu;\rho\sigma}$ introduced earlier

$$\begin{aligned}\boldsymbol{D}_{0,\gamma_{x0}} &\equiv \sum_{n=0}^{\infty}\frac{1}{(2d)^n}\sum_{\gamma\in\Gamma_n(0,x)} 1\\ \boldsymbol{D}_{4,\gamma_{x0}}^{\mu\nu;\rho\sigma} &\equiv \sum_{n=0}^{\infty}\frac{1}{(2d)^n}\sum_{\gamma\in\Gamma_n(0,x)} A_{\mu\nu}(\gamma_{x0}\otimes\gamma)A_{\rho\sigma}(\gamma_{x0}\otimes\gamma)\,.\end{aligned} \tag{3.5}$$

The coefficient $\boldsymbol{D}_{0,\gamma_{x0}}$ depends only on the points $0, x$ but not on the path γ_{x0} chosen to connect them. However, the coefficient $\boldsymbol{D}_{4,\gamma_{x0}}$ a priori depends on the path γ_{x0} as well, since the projected areas $A_{\mu\nu}(\gamma_{x0}\otimes\gamma)$ depend on the shape of the closed contour $\gamma_{x0}\otimes\gamma$.

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From eq. (3.4), we see that the separation between the operators ϕ^* and ϕ does not change the ultraviolet behavior of the expectation value (we obtain the same powers of a in the expansion as in eq. (2.17)), provided that the separation is a fixed number of lattice spacings (i.e. it shrinks as we take the limit $a \to 0$). Note that, although the ultraviolet divergences in operators with a separation have the same strength as in local operators, the coefficients have different numerical values. Therefore, the ultraviolet divergences do not cancel when we combine these operators, e.g. to form derivatives. In contrast, when the continuum limit is taken in this fashion for non-local operators, we get exactly the same infrared behavior (see the section 3.3.3) as for local operators. Since operators with derivatives involve differences between operators with different separations, the presence of a derivative leads to the cancellation of the leading infrared divergence.

3.3 Bilocal operators with a 1-hop separation

Let us first consider the specific case where the points 0 and x are nearest neighbors on the lattice and the path γ_{x0} that connects them is the shortest possible, i.e. an elementary link on the cubic lattice.

3.3.1 Coefficient $\boldsymbol{D}_{0,\hat{x}}$

For the coefficient $\boldsymbol{D}_{0,\gamma_{x0}}$, we can without loss of generality assume that the link γ_{x0} is a link $\gamma_{x0} = \hat{x}$. The random walks that connects 0 to $x = 0 + \hat{x}$ must have an odd length, $n = 2m + 1$. Simple combinatorics leads to the following expression,

$$\boldsymbol{D}_{0,\hat{x}} = \sum_{m=0}^{\infty} \frac{1}{(2d)^{2m+1}} \sum_{n_1+\cdots+n_d=m} \frac{(2m+1)!}{n_1!(1+n_1)!n_2!^2 \cdots n_d!^2} . \tag{3.6}$$

By a Borel transformation similar to the one used in the section 2.6, we can derive the following integral representation for this coefficient[10]

$$\boldsymbol{D}_{0,\hat{x}} = \int_0^{\infty} dt \; e^{-t} \, I_1\left(\frac{t}{d}\right) I_0^{d-1}\left(\frac{t}{d}\right) . \tag{3.7}$$

In 3 dimensions, this leads to a closed expression ([39] — section 6.612.6),

$$\boldsymbol{D}_{0,\hat{x}} \underset{d=3}{=} \frac{\sqrt{3}-1}{32\pi^3} \Gamma^2\left(\frac{1}{24}\right) \Gamma^2\left(\frac{11}{24}\right) - 1 \approx 0.51638606 , \tag{3.8}$$

while in 4 dimensions we get

$$\boldsymbol{D}_{0,\hat{x}} \underset{d=4}{\approx} 0.2394671218 . \tag{3.9}$$

[10] In the appendix B, we derive an integral expression for this leading order coefficient $\boldsymbol{D}_{0,\gamma_{x0}}$ when 0 and x are not nearest neighbors.

3.3.2 Coefficient $\boldsymbol{D}^{12;12}_{4,\hat{\sigma}}$

We must extend the discussion of the subsection 2.7 to a summation over all closed loops with a fixed link, of the type shown in the figure 5. We need to consider two cases:

a. The link γ_{x0} connecting the two points is not in the plane $\mu\nu$ on which we project the areas. The archetype of this case is $\mu\nu = 12$ and $\gamma_{x0} = \hat{3}$ (one elementary link in the $+x_3$ direction).

b. The link γ_{x0} lies in the $\mu\nu$ plane. For instance, $\mu\nu = 12$ and $\gamma_{x0} = \hat{1}$.

The first case **a** is the simplest, because the fixed link does not affect in any way the projected area in the 12 plane. The fixed link $\gamma_{x0} = \hat{3}$ only alters the counting of the hops that are orthogonal to the 12 plane. We now have

$$\begin{aligned}\boldsymbol{D}^{12;12}_{4,\hat{3}} \equiv & \sum_{m,n=0}^{\infty} \frac{1}{(2d)^{2(n+m)+1}} \binom{2(m+n)+1}{2m+1} \\ & \times \left[\sum_{\sigma\in\Gamma_{2m+1}(0,\hat{3})} 1\right]_{\text{dim } d-2} \times \left[\sum_{\gamma\in\Gamma_{2n}(0,0)} (A_{12}(\gamma))^2\right]_{\text{dim } 2} . \end{aligned} \tag{3.10}$$

Note that the number of hops in the transverse directions must be odd (we denote it $2m+1$). We can still use the formula (2.26) for the variances of the areas in dimension 2 (the second factor on the second line). The first factor on the second line counts the number of random walks of length $2m+1$ in the transverse directions that connect the points 0 and $\hat{3}$. In $d = 3$ and $d = 4$, this factor is given by

$$\begin{aligned}\left[\sum_{\sigma\in\Gamma_{2m+1}(0,\hat{3})} 1\right]_{\text{dim } 1} &= \binom{2m+1}{m} \\ \left[\sum_{\sigma\in\Gamma_{2m+1}(0,\hat{3})} 1\right]_{\text{dim } 2} &= \sum_{p+q=m} \frac{(2m+1)!}{p!(p+1)!q!^2} = \binom{2m+1}{m}^2 . \end{aligned} \tag{3.11}$$

Therefore, we find the following expressions for the coefficient $\boldsymbol{D}^{12;12}_{4,\hat{3}}$,

$$\boldsymbol{D}^{12;12}_{4,\hat{3}} \underset{d=3}{=} \sum_{l=0}^{\infty} \frac{(2l+1)!}{6^{2l+1}} \sum_{n=0}^{l} \frac{(2n)!}{(l-n)!(l-n+1)!n!^4} \frac{n^2(n-1)}{6(2n-1)} , \tag{3.12}$$

and

$$\boldsymbol{D}^{12;12}_{4,\hat{3}} \underset{d=4}{=} \sum_{l=0}^{\infty} \frac{(2l+1)!}{8^{2l+1}} \sum_{n=0}^{l} \frac{(2(l-n)+1)!(2n)!}{(l-n)!^2(l-n+1)!^2n!^4} \frac{n^2(n-1)}{6(2n-1)} . \tag{3.13}$$

In the case **b**, the formula (2.26) must be modified in order to sum only over loops that start at the point 0 and end at the point $\hat{1}$. Let us consider random walks of length $2n-1$, so that the length is $2n$ after adding a hop $\hat{1}^{-1}$ to return to the starting point and close the loop. Consider the set of all the closed random walks of length $2n$, over which the sum in eq. (2.26) is performed. After $2n-1$ steps, these walks must be at one of the 4 four points $\hat{1}$, $-\hat{1}$, $\hat{2}$ or $-\hat{2}$ (since at the next hop they return at 0). Thus the set $\Gamma_{2n}(0,0)$ of these closed random walks can be partitioned into four subsets according to the point reached at the step $2n-1$. These subsets are identical up to rotations of angles multiple of $\pi/2$, and therefore each of these subsets has the same contribution to the variance of the area (since the area is invariant under these rotations). Since we are now interested only in the random walks that reach the point $\hat{1}$ after $2n-1$ steps, we see that eq. (2.26) must be replaced by[11]

$$\left[\sum_{\gamma\in\Gamma_{2n-1}(0,\hat{1})}(A_{12}(\gamma))^2\right]_{\text{dim }2}=\frac{1}{4}\binom{2n}{n}^2\frac{n^2(n-1)}{6(2n-1)}. \tag{3.14}$$

Note that the combinatorial prefactor can also be written as

$$\frac{1}{4}\binom{2n}{n}^2=\binom{2n-1}{n}^2, \tag{3.15}$$

which is the number of 2-dimensional random walks of length $2n-1$ from 0 to $\hat{1}$ (see the second of eqs. (3.11)). From this, it is straightforward to find the analogue of eqs. (3.12) and (3.13) in the case where $\gamma_{x0}=\hat{1}^{-1}$:

$$\boldsymbol{D}^{12;12}_{4,\hat{1}}\underset{d=3}{=}\sum_{l=0}^{\infty}\frac{(2l+1)!}{6^{2l+1}}\sum_{n=0}^{l}\frac{(2n+1)!}{(l-n)!^2n!^2(n+1)!^2}\frac{(n+1)^2n}{6(2n+1)}, \tag{3.16}$$

and

$$\boldsymbol{D}^{12;12}_{4,\hat{1}}\underset{d=4}{=}\sum_{l=0}^{\infty}\frac{(2l+1)!}{8^{2l+1}}\sum_{n=0}^{l}\frac{(2(l-n))!(2n+1)!}{(l-n)!^4n!^2(n+1)!^2}\frac{(n+1)^2n}{6(2n+1)}. \tag{3.17}$$

(Note that, compared to eq. (3.14), we have shifted $n\to n+1$, so that the number of hops in the 12 plane is $2n+1$ instead of $2n-1$.)

The combinatorial formulas (3.12), (3.13), (3.16) and (3.17) can be transformed into an integral by means of a Borel transformation. We find the following results, valid in d dimensions:

$$\boldsymbol{D}^{12;12}_{4,\hat{3}}=\frac{1}{12d^2}\int_0^{\infty}dt\; e^{-t}\,t^2\,I_0^{d-3}\left(\frac{t}{d}\right)I_1^3\left(\frac{t}{d}\right), \tag{3.18}$$

and

$$\boldsymbol{D}^{12;12}_{4,\hat{1}}=\frac{1}{12d^2}\int_0^{\infty}dt\; e^{-t}\,t^2\,I_0^{d-1}\left(\frac{t}{d}\right)I_1\left(\frac{t}{d}\right). \tag{3.19}$$

3.3.3 Behavior for long random walks

Having in mind the infrared divergences that manifest themselves in the behavior of the summand in $\boldsymbol{C}_4^{12,12}$ for large path lengths $2l$, it is interesting to study also the summand in the newly introduced coefficients $\boldsymbol{D}^{12,12}_{4,\hat{1}}$ and $\boldsymbol{D}^{12,12}_{4,\hat{3}}$. In the figure 6, we plot as a function of the index l the relative differences between these quantities. It appears that these

[11]In order to obtain this formula, one may also use eq. (C.2) from the appendix C and sum over $1\leq n_1\leq n$,

$$\sum_{n_1+n_2=n}\frac{(2(n_1+n_2)-1)!}{n_1!(n_1-1)!n_2!^2}\frac{n_1n_2}{3}=\frac{1}{4}\binom{2n}{n}^2\frac{n^2(n-1)}{6(2n-1)}.$$

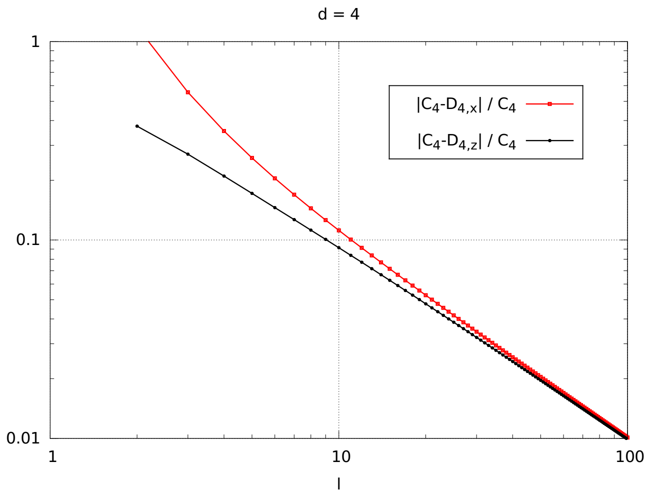

Figure 6. Relative difference between the summands in the coefficients $\boldsymbol{C}_4^{12,12}$, $\boldsymbol{D}_{4,\hat{1}}^{12,12}$ and $\boldsymbol{D}_{4,\hat{3}}^{12,12}$, as a function of the length of the random walk, in $d = 4$ dimensions.

differences decrease as l^{-1} when $l \to \infty$. In other words, the coefficients $\boldsymbol{C}_4^{12,12}$, $\boldsymbol{D}_{4,\hat{1}}^{12,12}$ and $\boldsymbol{D}_{4,\hat{3}}^{12,12}$ all have the same behavior for large random walks. This is in fact quite intuitive, since large values of l correspond to long random walks, that depend very little about the conditions that are enforced at the endpoints of the path.

This observation has also very practical consequences in the evaluation of the expectation of local operators that contain derivatives. In the particular example of eq. (3.1), it is easy to see that the second order coefficients in the small a expansion will always come in combinations such as $\boldsymbol{D}_{4,\hat{1}}^{12,12} - \boldsymbol{C}_4^{12,12}$ or $\boldsymbol{D}_{4,\hat{3}}^{12,12} - \boldsymbol{C}_4^{12,12}$. The summands in these differences have a milder behavior at large l, and instead of having a quadratic infrared divergence, they have only a logarithmic divergence

$$\boldsymbol{D}_{4,\hat{1}}^{12,12} - \boldsymbol{C}_4^{12,12} \sim \boldsymbol{D}_{4,\hat{3}}^{12,12} - \boldsymbol{C}_4^{12,12} \sim \log(l_{\max})\,, \tag{3.20}$$

as expected on dimensional grounds.

3.4 Bilocal operators with a 2-hop separation

In the discretization of $(D_\nu\phi)^*(D_\mu\phi)$, that appears for instance in the energy-momentum tensor, we also need to consider operators such as

$$\phi_a^*(x+\hat{\nu})U_\nu^{ab\dagger}(x)U_\mu^{bc}(x)\phi_c(x+\hat{\mu})\,. \tag{3.21}$$

When $\mu = \nu$, the two Wilson lines cancel and this operator is simply the local operator $\phi^*\phi$ evaluated at the point $x + \hat{\mu}$. The novel case is when $\mu \neq \nu$, for which the two elementary fields are separated by two hops in distinct directions.

For the leading order coefficient $\boldsymbol{D}_{0,\hat{\nu}^{-1}\hat{\mu}}$, we can use the result derived in the appendix B,

$$\boldsymbol{D}_{0,\hat{\nu}^{-1}\hat{\mu}} = \int_0^\infty dt\; e^{-t}\; I_1^2\Big(\frac{t}{d}\Big) I_0^{d-2}\Big(\frac{t}{d}\Big)\,. \tag{3.22}$$

For the next-to-leading order coefficient $\boldsymbol{D}^{12;12}_{4,\hat{\nu}^{-1}\hat{\mu}}$, we need now to distinguish three cases, depending on whether the directions μ, ν coincide with the directions $1, 2$ of the plane on which the areas are projected. The results for a single hop separation can be generalized into:

i. $\mu, \nu \neq 1, 2$:

$$\boldsymbol{D}^{12;12}_{4,\hat{\nu}^{-1}\hat{\mu}} = \frac{1}{12d^2} \int_0^\infty dt\, e^{-t}\, t^2\, I_1^4\left(\frac{t}{d}\right)\, I_0^{d-4}\left(\frac{t}{d}\right), \tag{3.23}$$

ii. $\nu = 1, \mu \neq 1, 2$:

$$\boldsymbol{D}^{12;12}_{4,\hat{1}^{-1}\hat{\mu}} = \frac{1}{12d^2} \int_0^\infty dt\, e^{-t}\, t^2\, I_1^2\left(\frac{t}{d}\right)\, I_0^{d-2}\left(\frac{t}{d}\right), \tag{3.24}$$

iii. $\nu = 1, \mu = 2$: by using eq. (C.4) and a Borel transformation, we can obtain the following expression:

$$\boldsymbol{D}^{12;12}_{4,\hat{1}^{-1}\hat{2}} = \frac{1}{6} \int_0^\infty dt\, e^{-t}\, I_0^{d-2}\left(\frac{t}{d}\right) \left[\frac{t^2}{2d^2} I_0^2\left(\frac{t}{d}\right) + I_1^2\left(\frac{t}{d}\right)\right]. \tag{3.25}$$

4 Anisotropic lattice

4.1 Worldline representation

Let us now consider a lattice in which the spacings are not the same in all directions: $a_1, a_2, \cdots, a_d$. The covariant derivative squared now reads

$$\begin{aligned} -D^2 f(i_1, \cdots, i_d) &= -2\left[\sum_{r=1}^d \frac{1}{a_r^2}\right] f(i_1, \cdots, i_d) \qquad (4.1)\\ &+ \sum_{r=1}^d \left\{ \frac{U_r(\cdots, i_r, \cdots) f(\cdots, i_r+1, \cdots)}{a_r^2} + \frac{U_r^{-1}(\cdots, i_r-1, \cdots) f(\cdots, i_r-1, \cdots)}{a_r^2} \right\}. \end{aligned}$$

Let us introduce a "mean inverse squared lattice spacing",

$$\frac{1}{\mathbf{a}^2} \equiv \frac{1}{d} \sum_{r=1}^d \frac{1}{a_r^2}, \tag{4.2}$$

and now we write

$$\frac{1}{D^2} \equiv \frac{\mathbf{a}^2}{2d} \sum_{n=0}^\infty (1 - \mathbf{a}^2 D^2/2d)^n. \tag{4.3}$$

This has again the virtue of eliminating the "stationary" term in the random walk, since we have

$$\begin{aligned} \left(1 - \frac{\mathbf{a}^2 D^2}{2d}\right) f(i_1, \cdots, i_d) = \frac{\mathbf{a}^2}{2d} \sum_{r=1}^d \Bigg\{ &\frac{U_r(\cdots, i_r, \cdots) f(\cdots, i_r+1, \cdots)}{a_r^2} \\ &+ \frac{U_r^{-1}(\cdots, i_r-1, \cdots) f(\cdots, i_r-1, \cdots)}{a_r^2} \Bigg\}. \qquad (4.4) \end{aligned}$$

The difference compared to the isotropic case is that the probability to make a hop in the direction r is modified by the factor

$$h_r \equiv \frac{\mathbf{a}^2}{a_r^2}\,. \tag{4.5}$$

This factor is trivially 1 if the spacings are all equal but differs from 1 for an anisotropic lattice. The hopping probability in the direction r is inversely proportional to the squared lattice spacing in this direction (the factor h_r is almost equal to d in the direction that has the smallest lattice spacing). Eq. (2.11) is thus generalized into

$$\langle\phi_a(0)\phi_a^*(0)\rangle = \frac{\mathbf{a}^2}{2d\prod_r a_r}\sum_{n=0}^{\infty}\frac{1}{(2d)^{2n}}\sum_{\gamma\in\Gamma_{2n}(0,0)}\left[\prod_{\ell\in\gamma}h_\ell\right]\mathrm{tr}_{\mathrm{adj}}\left(\mathcal{W}(\gamma)\right), \tag{4.6}$$

where $\prod_{\ell\in\gamma}h_\ell$ denotes the product of all the h's collected along the closed loop.

4.2 Geometrical interpretation

Note that the mean squared value of the absolute distance traveled in the direction r is given by

$$\langle\Delta\ell_r^2\rangle = n_r\, a_r^2\,, \tag{4.7}$$

where n_r is the number of hops in the direction r. Since n_r is proportional to h_r, the product $n_r a_r^2$ is independent of the direction r, and we have

$$\langle\Delta\ell_1^2\rangle = \langle\Delta\ell_2^2\rangle = \cdots = \langle\Delta\ell_d^2\rangle\,, \tag{4.8}$$

regardless of the values of the lattice spacings. In other words, the loops that enter in the formula (4.6) have an absolute geometrical shape which is isotropic (on average). On an anisotropic lattice, this isotropic distribution of shapes is realized by making more hops in the directions that have a smaller lattice spacing.

4.3 Leading order coefficients $\boldsymbol{C}_0$ and $\boldsymbol{D}_{0,\hat{\boldsymbol{x}}}$

The coefficient that appears in the leading term of the continuum limit is now modified into

$$\boldsymbol{C}_0(\{h_r\}) \equiv \sum_{n=0}^{\infty}\frac{1}{(2d)^{2n}}\sum_{\gamma\in\Gamma_{2n}(0,0)}\left[\prod_{\ell\in\gamma}h_\ell\right]. \tag{4.9}$$

A somewhat more explicit expression can be obtained by partitioning the $2n$ hops as $2n = 2n_1+\cdots+2n_d$, where $2n_1$ is the number of hops in the x_1 direction, $2n_2$ the number of hops in the x_2 direction, etc... This leads to

$$\boldsymbol{C}_0(\{h_r\}) = \sum_{n=0}^{\infty}\frac{(2n)!}{(2d)^{2n}}\sum_{n_1+\cdots+n_d=n}\frac{h_1^{2n_1}\cdots h_d^{2n_d}}{n1!^2\cdots n_d!^2}\,. \tag{4.10}$$

One can easily generalize the derivation of eq. (2.22) in the anisotropic case. This leads to the following integral representation for $\boldsymbol{C}_0$,

$$\boldsymbol{C}_0(\{h_r\}) = \int_0^\infty dt\; e^{-t}\prod_{r=1}^{d} I_0\left(\frac{h_r t}{d}\right). \tag{4.11}$$

Similarly to eq. (4.11), it is possible to obtain the following integral representation of the leading order coefficient $\boldsymbol{D}_{0,\hat{x}}$ that appears in the continuum expansion of composite operators involving fields separated by one lattice spacing:

$$\boldsymbol{D}_{0,\hat{1}}(\{h_r\}) = \int_0^\infty dt\, e^{-t}\, I_1\left(\frac{h_1 t}{d}\right) \prod_{i\neq 1} I_0\left(\frac{h_i t}{d}\right). \tag{4.12}$$

Likewise, the anisotropic form of the formulas of the appendix B is completely straightforward.

In the example of the operator of eq. (3.1), the leading term in the expansion in powers of the lattice coupling can be expressed in terms of the coefficients $\boldsymbol{C}_0$ and $\boldsymbol{D}_{0,\hat{x}}$,

$$\langle \phi_a^*(0)(D_\mu D_\mu \phi)_a(0)\rangle = \frac{\mathrm{tr}_{\mathrm{adj}}\,(1)}{d}\,\frac{\mathbf{a}^2}{a_1\cdots a_d}\sum_{i=1}^d \frac{\boldsymbol{D}_{0,\hat{\imath}} - \boldsymbol{C}_0}{a_i^2} + \cdots \tag{4.13}$$

Using the identities $I_1 = I_0'$ and $(xI_1(x))' = xI_0(x)$, and integrating by parts in eq. (4.12), we obtain

$$\sum_{i=1}^d h_i\, \boldsymbol{D}_{0,\hat{\imath}} = d\left(\boldsymbol{C}_0 - 1\right). \tag{4.14}$$

This identity leads to

$$\sum_{i=1}^d \frac{\boldsymbol{D}_{0,\hat{\imath}} + 1 - \boldsymbol{C}_0}{a_i^2} = 0\,. \tag{4.15}$$

When we use this property in eq. (4.13), it turns it into

$$\langle \phi_a^*(0)(D_\mu D_\mu \phi)_a(0)\rangle = -\frac{\mathrm{tr}_{\mathrm{adj}}\,(1)}{a_1\cdots a_d} + \cdots\,, \tag{4.16}$$

which is consistent with the equation of motion satisfied by the propagator:

$$\langle \phi_a^*(0)(D_\mu D_\mu \phi)_a(0)\rangle = \lim_{x\to 0}(D_\mu D_\mu)_x^{ab}\langle \phi_a^*(0)\phi_b(x)\rangle = -\mathrm{tr}_{\mathrm{adj}}\,(1)\,\lim_{x\to 0}\delta(x)\,. \tag{4.17}$$

A similar identity, eq. (4.24), among the coefficients that appear at the next order ensures that this property does not receive corrections that depend on the background field (i.e. the dots in eq. (4.16) are in fact zero).

4.4 Coefficient $\boldsymbol{C}_4^{12;12}$

In the isotropic case, an essential ingredient in the calculation of the coefficient $\boldsymbol{C}_4$ was the combinatorial formula (2.26), that gives the variance of the areas enclosed by closed random walks in 2 dimensions. Now, we need to generalize this formula to random walks weighted by the factors h_i that depend on the direction of each hop. Note that in the isotropic case ($h_i = 1$), this quantity has a simple generating function,

$$\sum_{n=0}^\infty \frac{X^{2n}}{(2n)!}\left[\sum_{\gamma\in\Gamma_{2n}(0,0)} \left(A_{12}(\gamma)\right)^2\right]_{\mathrm{dim}\ 2} = \frac{X^2}{3}\, I_1^2(2X)\,. \tag{4.18}$$

The integral representation (4.11) of $\boldsymbol{C}_0$ in the anisotropic case and its comparison with the corresponding isotropic formula (2.34) suggests the following generalization of eq. (4.18):

$$\sum_{n=0}^{\infty} \frac{X^{2n}}{(2n)!} \left[\sum_{\gamma \in \Gamma_{2n}(0,0)} \left[\prod_{l\in\gamma} h_l \right] \, (A_{12}(\gamma))^2 \right]_{\text{dim } 2} = \frac{h_1 h_2 X^2}{3} \, I_1(2h_1 X)\, I_1(2h_2 X)\,. \tag{4.19}$$

It is straightforward to see that this formula is equivalent to the identity (C.1) listed in the appendix C. In a sense, eq. (C.1) can be viewed as a "fine grained" version of eq. (2.26), that retains the information about the number of hops in each direction. We present a proof of this identity in a separate paper [38].

With the help of eq. (4.19), it is immediate to obtain the integral representation of the coefficient $\boldsymbol{C}_4^{12;12}$ in the anisotropic case,

$$\boldsymbol{C}_4^{12;12}(\{h_r\}) = \frac{h_1 h_2}{12 d^2} \int_0^\infty dt \; e^{-t}\, t^2 \left[\prod_{i\neq 1,2} I_0\!\left(\frac{h_i t}{d}\right) \right] I_1\!\left(\frac{h_1 t}{d}\right) I_1\!\left(\frac{h_2 t}{d}\right). \tag{4.20}$$

This coefficient will enter in the short distance expansion via the combination

$$a_1^2 a_2^2 \, \boldsymbol{C}_4^{12;12} F_a^{12}(0) F_a^{12}(0) \sim \mathbf{a}^4 F_a^{12}(0) F_a^{12}(0)\,, \tag{4.21}$$

where the unwritten factors are dimensionless numbers. Therefore, the terms in F^2 are accompanied by the fourth power of the "average lattice spacing" $\mathbf{a}$, with a numerical prefactor that depends on the ratios a_i/a_j of the lattice spacings in the various directions. Thus, if we take the short distance limit while keeping these ratios fixed, only the factor $\mathbf{a}^4$ decreases, while the numerical prefactor stays constant.

4.5 Coefficients $\boldsymbol{D}_{4,\hat{\mu}}^{12;12}$ and $\boldsymbol{D}_{4,\hat{\nu}^{-1}\hat{\mu}}^{12;12}$

Likewise, eqs. (3.18) and (3.19) can be generalized into

$$\boldsymbol{D}_{4,\hat{3}}^{12;12}(\{h_r\}) = \frac{h_1 h_2}{12 d^2} \int_0^\infty dt \; e^{-t}\, t^2 \left[\prod_{i\neq 1,2,3} I_0\!\left(\frac{h_i t}{d}\right) \right] I_1\!\left(\frac{h_1 t}{d}\right) I_1\!\left(\frac{h_2 t}{d}\right) I_1\!\left(\frac{h_3 t}{d}\right), \tag{4.22}$$

and

$$\boldsymbol{D}_{4,\hat{1}}^{12;12}(\{h_r\}) = \frac{h_1 h_2}{12 d^2} \int_0^\infty dt \; e^{-t}\, t^2 \left[\prod_{i\neq 2} I_0\!\left(\frac{h_i t}{d}\right) \right] I_1\!\left(\frac{h_2 t}{d}\right). \tag{4.23}$$

Using these formulas, we can also prove the identity[12]

$$\sum_{i=1}^{d} h_i \boldsymbol{D}_{4,\hat{\imath}}^{12;12} = d\, \boldsymbol{C}_4^{12;12}\,, \tag{4.24}$$

that generalizes eq. (4.15) to the coefficients that appear in the next order of the expansion.

It is also easy to generalize to the case an anisotropic lattice the coefficients $\boldsymbol{D}_{4,\hat{\nu}^{-1}\hat{\mu}}^{12;12}$ with a separation of two hops, given in eqs. (3.23), (3.24) and (3.25) in the isotropic case. One obtains:

[12] A word of caution is necessary here: the l.h.s. and r.h.s. of this equation contain infrared divergences that should be properly regularized, for instance by the introduction of a mass (see the section 2.8).

i. $\mu,\nu\neq 1,2$:

$$\boldsymbol{D}^{12;12}_{4,\hat{\nu}^{-1}\hat{\mu}}=\frac{h_1h_2}{12d^2}\int_0^\infty dt\,e^{-t}\,t^2\,I_1\left(\frac{h_1t}{d}\right)I_1\left(\frac{h_2t}{d}\right)I_1\left(\frac{h_\mu t}{d}\right)I_1\left(\frac{h_\nu t}{d}\right)\prod_{i\neq 1,2,\mu,\nu}I_0\left(\frac{h_it}{d}\right), \tag{4.25}$$

ii. $\nu=1,\mu\neq 1,2$:

$$\boldsymbol{D}^{12;12}_{4,\hat{1}^{-1}\hat{\mu}}=\frac{h_1h_2}{12d^2}\int_0^\infty dt\;e^{-t}\;t^2\;I_0\left(\frac{h_1t}{d}\right)I_1\left(\frac{h_2t}{d}\right)I_1\left(\frac{h_\mu t}{d}\right)\prod_{i\neq 1,2,\mu}I_0\left(\frac{h_it}{d}\right), \tag{4.26}$$

iii. $\nu=1,\mu=2$:

$$\boldsymbol{D}^{12;12}_{4,\hat{1}^{-1}\hat{2}}=\frac{1}{6}\int_0^\infty dt\;e^{-t}\left[\frac{h_1h_2t^2}{2d^2}I_0\left(\frac{h_1t}{d}\right)I_0\left(\frac{h_2t}{d}\right)+I_1\left(\frac{h_1t}{d}\right)I_1\left(\frac{h_2t}{d}\right)\right]\prod_{i\neq 1,2}I_0\left(\frac{h_it}{d}\right). \tag{4.27}$$

4.6 Limit of extreme anisotropy

An extreme case of anisotropy is to have one lattice spacing much smaller than the others. This could correspond to the case where one of the coordinates (e.g. the time) is treated as a continuous variable while the other are discretized with a small but finite lattice spacing. Let us therefore assume that

$$a_d\ll a_1,\cdots,a_{d-1}\to 0\,. \tag{4.28}$$

In order to get a sense of the behavior of the coefficients in this limit, let us consider the coefficient $\boldsymbol{C}_0$, whose integral representation (4.11) can easily be studied numerically. In this numerical study, we consider the dimension $d=4$, and the remaining 3 lattice spacings are assumed to be all equal. As shown in the figure 7, we see that $\boldsymbol{C}_0$ diverges as a_4^{-1} in this limit. In this limit, we have

$$\mathbf{a}^2\approx 4a_4^2\,,\qquad\qquad \frac{\mathbf{a}^2}{a_1a_2a_3a_4}\approx\frac{4a_4}{a_1a_2a_3}\,, \tag{4.29}$$

so that the leading term of eq. (4.6) in fact becomes independent of the smallest lattice spacing. This result is consistent with the result of the appendix F, where we use from the start a continuous time variable.

It is in fact possible to understand this limit analytically, starting from the integral representation of the coefficients that appear in the small lattice spacing expansion. Generically, these coefficients involve integrals of the form,

$$A_{n;\{\delta_i\}}\equiv\int_0^\infty dt\;e^{-t}\;t^n\;\prod_{i=1}^d I_{\delta_i}\left(\frac{h_it}{d}\right), \tag{4.30}$$

where the exponent n is 0 or 2 and the indices δ_i are 0 or 1, in all the examples we have encountered so far. In this limit, we have

$$h_d=d\,,\qquad\quad h_i=\frac{da_d^2}{a_i^2}\ll 1\quad (i<d)\,. \tag{4.31}$$

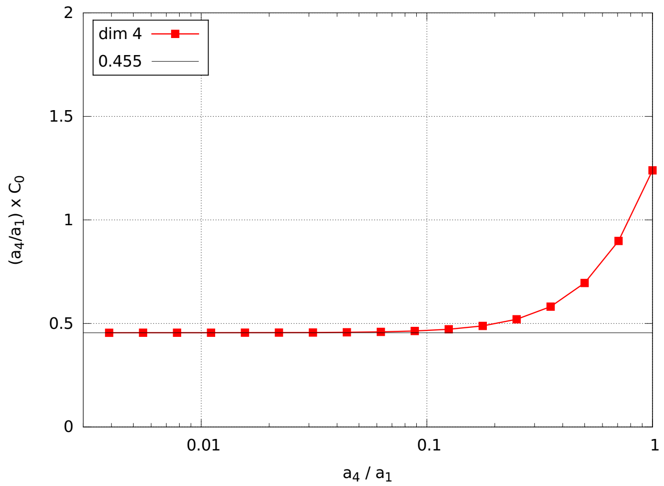

Figure 7. Behavior of the leading coefficient $\boldsymbol{C}_0$ as a function of the ratio of lattice spacings a_4/a_1, in 4 dimensions.

Therefore, the argument of the last Bessel function is much larger than the arguments of the first $d-1$ Bessel functions. Let us first define a rescaled integration variable by

$$\tau = t\left(1 - \frac{h_d}{d}\right) = t\,\frac{d_s a_d^2}{\mathbf{a}_s^2 + d_s a_d^2}\,, \tag{4.32}$$

where we denote

$$d_s \equiv d-1\,, \qquad\qquad \frac{1}{\mathbf{a}_s^2} \equiv \frac{1}{d_s}\sum_{i=1}^{d_s}\frac{1}{a_i^2}\,. \tag{4.33}$$

Using $h_i t/d = \mathbf{a}_s^2\tau/(d_s a_i^2)$ and the asymptotic expansion of the modified Bessel function,

$$I_\delta(z) \underset{z\to+\infty}{=} \frac{e^z}{\sqrt{2\pi}}\left[\frac{1}{\sqrt{z}} + \mathcal{O}\big(z^{-3/2}\big)\right], \tag{4.34}$$

we arrive at

$$A_{n;\{\delta_i\}} \underset{a_d\to 0}{=} \frac{1}{\sqrt{2\pi}}\left(\frac{\mathbf{a}_s^2}{d_s a_d^2}\right)^{n+\frac{1}{2}} \int_0^\infty d\tau\; e^{-\tau}\;\tau^{n-1/2}\prod_{i=1}^{d_s} I_{\delta_i}\!\left(\frac{h_{is}\tau}{d_s}\right), \tag{4.35}$$

where we define $h_{is} \equiv \mathbf{a}_s^2/a_i^2$. Note that this limiting value does not depend on the index δ_d of the Bessel function associated to the temporal direction.

When $n = 0$ (i.e. for the coefficients $\boldsymbol{C}_0$ and $\boldsymbol{D}_0$), this integral behaves as a_d^{-1}, which is cancelled by the behavior of the overall prefactor $\mathbf{a}^2/(a_1\cdots a_d)$. Therefore, the leading order terms have a finite limit[13] when $a_d \to 0$. In the next-to-leading order coefficients

[13] If we take $a_1 = \cdots = a_{d-1} \gg a_d$, then the leading term of $\langle\phi_a^*(0)\phi_a(0)\rangle$ reads

$$\langle\phi_a^*(0)\phi_a(0)\rangle \underset{a_d\ll a_1\cdots a_{d-1}\to 0}{=} \frac{\mathrm{tr}_{\mathrm{adj}}\,(1)}{2a_1^{d-2}}\underbrace{\frac{1}{\sqrt{2\pi d_s}}\int_0^\infty \frac{d\tau}{\sqrt{\tau}}\;e^{-\tau}\;I_0^{d_s}\!\left(\frac{\tau}{d_s}\right)}_{0.4553440518\cdots\ \text{if}\ d_s=3} + \cdots\,,$$

in perfect agreement with the figure 7.

$\boldsymbol{C}_4^{ij;ij}$ and $\boldsymbol{D}_4^{ij;ij}$, the integral (4.35) appear with the exponent $n = 2$ and therefore it behaves as a_d^{-5}. One of the powers of a_d^{-1} is cancelled by the prefactor $\mathbf{a}^2/(a_1 \cdots a_d)$. Moreover, the coefficients $\boldsymbol{C}_4^{ij;ij}$ and $\boldsymbol{D}_4^{ij;ij}$ also contain a prefactor $h_i h_j$,

$$h_i h_j \underset{a_d \to 0}{=} \frac{d^2 a_d^4}{a_i^2 a_j^2}\,, \tag{4.36}$$

thereby canceling the remaining factor a_d^{-4} from eq. (4.35). Therefore, the next-to-leading order coefficients are also finite in the limit $a_d \to 0$ (aside from the possible infrared divergences discussed previously).

4.7 Energy-momentum tensor

Thanks to the results of the previous sections, we can write the short distance expansion of the energy momentum tensor. In this section, we present expressions for its diagonal components (its off-diagonal components can be treated similarly, but have a slightly more complicated structure). These diagonal components read:

$$T^{ii} = (D_i\phi)^*(D_i\phi) - \sum_{j\neq i}(D_j\phi)^*(D_j\phi)\,. \tag{4.37}$$

Therefore, we need the expression of the expectation value of the operator $(D_i\phi)^*(D_i\phi)$ (not summed over i). The lattice version of this operator is

$$(D_i\phi(0))^*(D_i\phi(0)) = \frac{\phi^*(0)\phi(0)+\phi^*(\hat\imath)\phi(\hat\imath)-\phi^*(\hat\imath)U_i^\dagger(0)\phi(0)-\phi^*(0)U_i(0)\phi(\hat\imath)}{a_i^2}\,, \tag{4.38}$$

and the worldline representation of its 1-loop expectation value reads

$$\begin{aligned}\langle (D_i\phi(0))^*(D_i\phi(0))\rangle = \frac{\mathbf{a}^2}{2d\prod_r a_r}\frac{1}{a_i^2}\sum_{n=0}^{\infty}\frac{1}{(2d)^n}\Bigg\{ &\sum_{\gamma\in\Gamma_n(0,0)}\Big[\prod_{\ell\in\gamma}h_l\Big]\,\mathrm{tr}_{\mathrm{adj}}\left(\mathcal{W}(\gamma)\right)\\ &+\sum_{\gamma\in\Gamma_n(\hat\imath,\hat\imath)}\Big[\prod_{\ell\in\gamma}h_l\Big]\,\mathrm{tr}_{\mathrm{adj}}\left(\mathcal{W}(\gamma)\right)\\ &-\sum_{\gamma\in\Gamma_n(\hat\imath,0)}\Big[\prod_{\ell\in\gamma}h_l\Big]\,\mathrm{tr}_{\mathrm{adj}}\left(U_i^\dagger(0)\mathcal{W}(\gamma)\right)\\ &-\sum_{\gamma\in\Gamma_n(0,\hat\imath)}\Big[\prod_{\ell\in\gamma}h_l\Big]\,\mathrm{tr}_{\mathrm{adj}}\left(U_i(0)\mathcal{W}(\gamma)\right)\Bigg\}\,.\end{aligned} \tag{4.39}$$

Using results derived earlier in this paper, we can write the following short distance expansion for this quantity:

$$\begin{aligned}\langle (D_i\phi(0))^*(D_i\phi(0))\rangle = \frac{\mathbf{a}^2}{2d\prod_r a_r}\frac{2}{a_i^2}\Bigg\{&\mathrm{tr}_{\mathrm{adj}}\,(1)\,\big(\boldsymbol{C}_0-\boldsymbol{D}_{0,\hat\imath}\big)\\ &-\frac{g^2}{4}\sum_{\mu<\nu}a_\mu^2 a_\nu^2\big(F_a^{\mu\nu}(0)\big)^2\,\big(\boldsymbol{C}_4^{\mu\nu;\mu\nu}-\boldsymbol{D}_{4,\hat\imath}^{\mu\nu;\mu\nu}\big)+\cdots\Bigg\}\,.\end{aligned} \tag{4.40}$$

Using eq. (4.37), as well as eqs. (4.15) and (4.24), we obtain the following expansion for the diagonal components of the energy-momentum tensor,

$$\begin{aligned} T^{ii} = \frac{\mathbf{a}^2}{d\prod_r a_r} \Bigg\{ &\mathrm{tr}_{\mathrm{adj}}\,(1) \left(\frac{2}{a_i^2} (\boldsymbol{C}_0 - \boldsymbol{D}_{0,\hat{\imath}}) - \frac{d}{\mathbf{a}^2} \right) \\ &- \frac{g^2}{2a_i^2} \sum_{\substack{k<l \\ k,l\neq i}} a_k^2 a_l^2 \left(F_a^{kl}(0)\right)^2 \left(\boldsymbol{C}_4^{kl;kl} - \boldsymbol{D}_{4,\hat{\imath}}^{kl;kl}\right) \\ &- \frac{g^2}{2a_i^2} \sum_{k\neq i} a_i^2 a_k^2 \left(F_a^{ik}(0)\right)^2 \left(\boldsymbol{C}_4^{ik;ik} - \boldsymbol{D}_{4,\hat{\imath}}^{ik;ik}\right) + \cdots \Bigg\}, \end{aligned} \tag{4.41}$$

where we have explicitly separated the terms of order a^4 depending on whether one of the indices carried by the field strength is i or not. Integral expressions in terms of modified Bessel functions for all the coefficients that appear in this formula, for arbitrary lattice spacings, can be found in the previous sections of this paper. By using eqs. (4.15) and (4.24), we can see that

$$\sum_{i=1}^{d} T^{ii} = (2-d) \frac{\mathrm{tr}_{\mathrm{adj}}\,(1)}{\prod_r a_r} + \cdots, \tag{4.42}$$

where the dots are UV finite terms in four dimensions.

Eq. (4.41) provides the final answer for generic lattice spacings. If in addition we assume that the temporal spacing a_d is much smaller than the others, we can use

$$\begin{aligned} \frac{2}{a_i^2}(\boldsymbol{C}_0 - \boldsymbol{D}_{0,\hat{\imath}}) - \frac{d}{\mathbf{a}^2} \underset{a_d\to 0}{=} &-\frac{1}{a_d^2} \\ &+ \left(\frac{2\,\mathbf{a}_s^2}{\pi d_s a_d^2}\right)^{1/2} \frac{1}{a_i^2} \int_0^\infty \frac{d\tau}{\sqrt{\tau}}\, e^{-\tau} \left[I_0\!\left(\frac{h_{is}\tau}{d_s}\right) - I_1\!\left(\frac{h_{is}\tau}{d_s}\right)\right] \prod_{j\neq i,d} I_0\!\left(\frac{h_{js}\tau}{d_s}\right) \\ \frac{2}{a_d^2}(\boldsymbol{C}_0 - \boldsymbol{D}_{0,\hat{d}}) - \frac{d}{\mathbf{a}^2} \underset{a_d\to 0}{=} &\frac{1}{a_d^2}, \end{aligned} \tag{4.43}$$

respectively for $i \neq d$ and $i = d$. For the next-to-leading order coefficients, we can also use eq. (4.35) to obtain their limiting value if $i \neq d$. If $i = d$, it is necessary to go one order further in the asymptotic expansion of the Bessel function of argument $h_d t/d$. In this case, we can use the following limit

$$\begin{aligned} &\frac{1}{a_d^2} \int_0^\infty dt\, e^{-t}\, t^2 \left[I_0\!\left(\frac{h_d t}{d}\right) - I_1\!\left(\frac{h_d t}{d}\right)\right] \prod_{i=1}^{d-1} I_{\delta_i}\!\left(\frac{h_i t}{d}\right) \\ &\underset{a_d\to 0}{=} \frac{1}{\sqrt{8\pi}} \frac{d_s}{\mathbf{a}_s^2} \left(\frac{\mathbf{a}_s^2}{d_s a_d^2}\right)^{5/2} \int_0^\infty d\tau\, e^{-\tau}\, \tau^{1/2} \prod_{i=1}^{d_s} I_{\delta_i}\!\left(\frac{h_{is}\tau}{d_s}\right). \end{aligned} \tag{4.44}$$

The terms $\pm a_d^{-2}$ in eq. (4.43) lead to contributions proportional to $(a_1 \cdots a_d)^{-1}$. It is easy to check that all the other contributions to T^{ii} become independent of a_d in the limit $a_d \to 0$.

5 Conclusions

In this paper, we have applied a discrete version of the worldline formalism in order to obtain expressions for 1-loop expectation values in a lattice scalar field theory, in the presence of a non-Abelian gauge background. In this framework, 2-point correlators are expressed as sums over all the random walks that connect their endpoints (or closed loops in the specific case of local operators). This representation renders the ultraviolet and infrared behaviors of these expectation values very intuitive: ultraviolet divergences are encoded in the contribution of very short random walks, while the infrared behavior arises from the statistics of long random walks that explore large regions of space-time. Moreover, this formalism has the virtue of organizing the calculation in such a way that only gauge invariant objects appear in intermediate steps.

Moreover, it is straightforward to take the limit of small lattice spacing in these worldline expressions. This gives an expansion in powers of the background field strength, whose coefficients are sums of all the closed random walks on the lattice, thereby reducing the calculation of the coefficients to a combinatorial problem. The leading coefficient is merely counting these random walks, and is therefore very easy to obtain. The next-to-leading order coefficient is related to the variance of the areas (projected on a plane) enclosed by these random walks. The relevant combinatorial formulas can be found in ref. [37]. Because of gauge invariance and locality, the terms that may appear in the small lattice spacing expansion is greatly constrained. For instance, in the massless case, the simplest such operator that can appear in the expansion is $F_a^{ij}F_a^{ij}$, that has dimension four. Therefore, the first subleading term in the continuum expansion is suppressed by four powers of the lattice spacing.

When one considers a lattice with anisotropic lattice spacings, these random walks are further weighted by factors that count the number of hops in each direction. In order to obtain simple expressions for the coefficients of the expansion in this case, we had to conjecture some generalizations of the formulas of ref. [37]. The proof of these formulas is given in a separate paper [38].

Using a Borel transformation, all the coefficients that appear in the short distance expansion of these correlators can be rewritten as 1-dimensional integrals. These can be easily evaluated numerically, and they are also quite convenient in order to study analytically the limit where one lattice spacing becomes much smaller than the others.

This work can be extended in several directions. One of them is to depart from a scalar field theory, in order to study quantities where a spin 1/2 fermion or a spin 1 gauge boson circulates in the loop. In the continuum theory, these extensions are well known. Another –considerably more difficult– extension would be to use the worldline formalism in order to go beyond one loop. Also, at the moment, it is unclear whether one could modify this formalism to handle a Minkowskian time (see the discussion in the appendix F).

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A Closed random walks on a periodic lattice

In the main body of this paper, all the combinatorial formulas that we have derived count random walks on an infinite cubic lattice. Since the lattice is infinite, there is no need to specify boundary conditions. This is the appropriate setting when one considers the limit $a \to 0$ at fixed physical volume: the constant volume is achieved by letting the number of lattice points go to infinity (as a^{-1}) in each direction. In this appendix, we consider a different limit, where we keep fixed the number of lattice points as we take the limit $a \to 0$. We now assume periodic boundary conditions.

We have introduced in eq. (2.4) a sequence of functions P_n defined on the lattice. If the first of these functions is

$$P_0(x) = \delta_{x,0}\,, \tag{A.1}$$

then $P_n(x)$ is the probability that a random walk starting at the point 0 reaches the point x after n steps.[14] With this choice of initial condition, $P_n(0)$ is the probability that the random walk makes a closed loop of base point 0 in n steps.

The first step is to find the spectrum of the linear operator that maps P_n to P_{n+1}. By using eq. (2.8), one finds that the eigenfunctions and eigenvalues of this operator are the plane waves

$$\phi_{\vec{k},d}(x) \equiv e^{2i\pi\frac{\{\vec{k}\cdot\vec{x}\}_d}{N}}\,, \qquad \{\vec{k}\cdot\vec{x}\}_d \equiv k_1x_1 + \cdots + k_dx_d\,, \tag{A.2}$$

and that the associated eigenvalue is

$$\Omega_{\vec{k},d} \equiv \frac{1}{d}\sum_{r=1}^{d}\cos\left(\frac{2\pi k_r}{N}\right)\,, \tag{A.3}$$

where N is the number of lattice spacings in each direction (for simplicity, we assume the same number of spacings in all the directions). The initial condition of the iteration can be written as

$$P_0(x) = \frac{1}{N^d}\sum_{\{\vec{k}\}_d}\phi_{\vec{k},d}(x)\,, \tag{A.4}$$

where $\{\vec{k}\}_d$ denotes all the d-uplets of integers in the range $[0, N-1]$. After n iterations this distribution has become

$$P_n(x) = \frac{1}{N^d}\sum_{\{\vec{k}\}_d}\Omega^n_{\vec{k},d}\;\phi_{\vec{k},d}(x)\,. \tag{A.5}$$

In order to obtain the probability for a random loop of length n to be closed, we simply evaluate this at $x = 0$,

$$P_n(0) = \frac{1}{N^d}\sum_{\{\vec{k}\}_d}\Omega^n_{\vec{k},d}\,. \tag{A.6}$$

[14] Using eq. (2.4) and "integrating by parts", it is easy to check that the quantity

$$\sum_{y\in\text{lattice}} P_n(y)$$

is conserved as n increases.

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The sum over n of the probability of closed random walks of length n therefore reads

$$\sum_{n\geq 0} P_n(0) = \frac{1}{N^d}\sum_{\{\vec{k}\}_d}\frac{1}{1-\Omega_{\vec{k},d}}\,, \tag{A.7}$$

where we recognize the lattice expression for a 1-loop tadpole.

Let us now consider in more detail eq. (A.6). We can make the right hand side more explicit by writing

$$\begin{aligned}\frac{1}{N^d}\sum_{\{\vec{k}\}_d}\Omega^n_{\vec{k},d} &= \frac{1}{N^d(2d)^n}\sum_{\{\vec{k}\}_d}\left(e^{2i\pi\frac{k_1}{N}}+e^{-2i\pi\frac{k_1}{N}}+\cdots+e^{2i\pi\frac{k_d}{N}}+e^{-2i\pi\frac{k_d}{N}}\right)^n\\ &= \frac{1}{N^d(2d)^n}\sum_{\{\vec{k}\}_d}\sum_{\substack{n_1+p_1+\cdots\\+n_d+p_d=n}}\frac{n!}{n_1!p_1!\cdots n_d!p_d!}\\ &\qquad\qquad\times e^{2i\pi\frac{k_1(n_1-d_1)}{N}}\cdots e^{2i\pi\frac{k_d(n_d-d_d)}{N}}\\ &= \frac{1}{(2d)^n}\sum_{\substack{n_1+p_1+\cdots\\+n_d+p_d=n}}\frac{n!}{n_1!p_1!\cdots n_d!p_d!}\delta_{n_1-p_1,0[N]}\cdots\delta_{n_d-p_d,0[N]}\,,\end{aligned} \tag{A.8}$$

where the symbol $\delta_{p,0[N]}$ is a Kronecker symbol that defines an equality modulo N:

$$\delta_{p,0[N]} \equiv \sum_{i\in\mathbb{Z}}\delta_{p,iN}\,. \tag{A.9}$$

The essential difference between an infinite lattice and a finite lattice with periodic boundary conditions comes from here.

If we take the limit $N\to\infty$, then we have

$$\delta_{p,0[N]} \underset{N\to\infty}{=} \delta_{p,0}\,, \tag{A.10}$$

and eq. (A.8) is equivalent to

$$\frac{1}{N^d}\sum_{\{\vec{k}\}_d}\Omega^n_{\vec{k},d} = \begin{cases}\dfrac{1}{(2d)^{2m}}\displaystyle\sum_{n_1+\cdots+n_d=m}\frac{(2m)!}{n_1!^2\cdots n_d!^2} & (n=2m \text{ even})\\ 0 & (n \text{ odd})\end{cases}\,, \tag{A.11}$$

which is nothing but the combinatorial expression for the coefficient $\boldsymbol{C}_0$ on an infinite cubic lattice.

In order to illustrate explicitly the difference on a finite periodic lattice, let us consider a 1-dimensional lattice of N sites. Eq. (A.8) becomes

$$\frac{1}{N}\sum_{\{\vec{k}\}_1}\Omega^n_{\vec{k},1} = \frac{1}{2^n}\sum_{p=0}^{n}\frac{n!}{p!(n-p)!}\delta_{2p,n[N]}\,. \tag{A.12}$$

For N even, this forces the random walk to have an even length $n=2m$, and we get

$$\frac{1}{N}\sum_{\{\vec{k}\}_1}\Omega^{2m}_{\vec{k},1} \underset{N\text{ even}}{=} \frac{1}{2^{2m}}\left[\frac{(2m)!}{m!^2}+2\frac{(2m)!}{(m+\frac{N}{2})!(m-\frac{N}{2})!}+2\frac{(2m)!}{(m+\frac{2N}{2})!(m-\frac{2N}{2})!}+\cdots\right], \tag{A.13}$$

where the sum in the right hand side stops when the argument of the second factorial in the denominator becomes negative. The first term of this formula is identical to the result on an infinite lattice. The additional terms corresponds to paths that wrap around the periodic lattice, with winding number ± 1 for the second term, ± 2 for the third term, etc... These terms can be viewed as finite size corrections, since they explicitly depend on N. When the lattice size N is odd, the situation is even more complicated because random walks of odd length are also permitted. For $n = 2m$, we get

$$\frac{1}{N}\sum_{\{\vec{k}\}_1}\Omega^{2m}_{\vec{k},1} \underset{N \text{ odd}}{=} \frac{1}{2^{2m}}\left[\frac{(2m)!}{m!^2} + 2\frac{(2m)!}{(m+N)!(m-N)!} + 2\frac{(2m)!}{(m+2N)!(m-2N)!} + \cdots\right], \tag{A.14}$$

and for odd lengths $n = 2m+1$ we have

$$\frac{1}{N}\sum_{\{\vec{k}\}_1}\Omega^{2m+1}_{\vec{k},1} \underset{N \text{ odd}}{=} \frac{1}{2^{2m}}\left[2\frac{(2m+1)!}{(m+\frac{1+N}{2})!(m+\frac{1-N}{2})!} + 2\frac{(2m+1)!}{(m+\frac{1+2N}{2})!(m+\frac{1-2N}{2})!} + \cdots\right]. \tag{A.15}$$

The situation in d dimensions follows the same pattern, but explicit formulas become quite cumbersome. In particular, since the random walk takes place on a d-dimensional torus instead of a ring, the "winding number" is now a d-uplet $(w_1, \cdots, w_d) \in \mathbb{Z}^d$. For a given length n of random walk, the term of winding number $(0, \cdots, 0)$ is identical to the result on an infinite lattice, but in addition one must sum over all the possible windings allowed for a given length n of the random walk.

B Coefficient $\boldsymbol{D}_{0,\gamma_{x0}}$ for arbitrary separations

In this appendix, we generalize the eqs. (3.6)–(3.9) to the case where the points 0 and x are separated by more than one hop. For definiteness, let us assume that the point x is

$$x \equiv x_1\hat{1} + \cdots + x_d\hat{d}\,. \tag{B.1}$$

The minimal length of a path connecting 0 to x is:

$$\Delta \equiv x_1 + \cdots + x_d\,. \tag{B.2}$$

Firstly, let us notice that the parity of the length of the paths γ_{x0} that connect the two points depends solely on the endpoints 0 and x. This parity is that of the number Δ. One can first obtain a combinatorial expression for $\boldsymbol{D}_{0,\gamma_{x0}}$,

$$\boldsymbol{D}_{0,\gamma_{x0}} = \sum_{n=\Delta}^{\infty}\frac{1}{(2d)^n}\sum_{\Delta+2(n_1+\cdots+n_d)=n}\frac{n!}{n_1!(n1+x_1)!\cdots n_d!(n_d+x_d)!}\,. \tag{B.3}$$

By a Borel transformation, this can be turned into

$$\boldsymbol{D}_{0,\gamma_{x0}} = \int_0^{\infty} dt\; e^{-t}\, B_d\left(\frac{t}{2d}\right), \tag{B.4}$$

$x_1x_2x_3$	r_1	r_2	r_3
000	1	0	0
100	1	0	-1/3
110	5/12	-1/2	0
200	10/3	-2	-2
111	-1/8	3/4	0
210	3/8	-9/4	1/3
300	35/2	21	-13

Table 2. Values of the coefficients $r_{1,2,3}$ for separations up to three hops (from [39] — section 6.612.6). These values are invariant under permutations of $x_{1,2,3}$. For larger separations, see [39] — section 6.612.6.

with

$$B_d(x) \equiv \prod_{i=1}^{d} \left[\sum_{p=0}^{\infty} \frac{x^{2p+x_i}}{p!(p+x_i)!} \right] = \prod_{i=1}^{d} I_{x_i}(2x)\,. \tag{B.5}$$

This leads to the following integral representation of $\boldsymbol{D}_{0,\gamma_{x0}}$,

$$\boldsymbol{D}_{0,\gamma_{x0}} = \int_0^\infty dt\; e^{-t} \prod_{i=1}^{d} I_{x_i}\left(\frac{t}{d}\right). \tag{B.6}$$

In 3 dimensions, this leads to a closed expression (see ref. [39] — section 6.612.6)

$$\boldsymbol{D}_{0,\gamma_{x0}} \underset{d=3}{=} 3\left(r_1 \mathfrak{g} + \frac{r_2}{\pi^2 \mathfrak{g}} + r_3\right), \tag{B.7}$$

with

$$\mathfrak{g} \equiv \frac{\sqrt{3}-1}{96\pi^3}\, \Gamma^2\left(\frac{1}{24}\right) \Gamma^2\left(\frac{11}{24}\right), \tag{B.8}$$

and where the coefficients $r_{1,2,3}$ are listed in the table 2 as a function of $x_{1,2,3}$ for small separations. In 4 dimensions, the integral of eq. (B.6) must be evaluated numerically. Some values for small separations are listed in the table 3.

C Moments of the distribution of areas in $d = 2$

C.1 Variance of the areas of closed loops with fixed sections

In the case of bilocal operators, the next-to-leading order coefficients $\boldsymbol{D}^{12;12}_{4,\gamma_{x0}}$ require that we evaluate the variance of the areas enclosed by the paths $\gamma \otimes \gamma_{x0}$, where γ is a 2-dimensional random walk from the point 0 to the point x. In other words, $\gamma \otimes \gamma_{x0}$ is a closed path, but only the section γ is random, while the section γ_{x0} is held fixed.

Having in mind the case of an anisotropic lattice, it is also useful to have formulas that keep track of the number of hops in the x_1 and x_2 directions separately. All the

$x_1x_2x_3x_4$	$\boldsymbol{D}_{0,x_{1,2,3,4}}$ $(d=4)$
0000	1.239467122
1000	0.2394671218
1100	0.1017176302
2000	0.06596407193
1110	0.06187238110
2100	0.04365863661
3000	0.02629363394

Table 3. Values of the leading order coefficient $\boldsymbol{D}_{0,\gamma_{x0}}$ in 4 dimensions, for separations up to three hops. These values are invariant under permutations of $x_{1,2,3,4}$.

Figure 8. Left: closed random walk from 0 to 0. Right: closed random walk with a fixed section of length 1 between 0 and x (shown in green).

Figure 9. Closed random walks with a fixed section of length 2 (shown in green). Left: $x = 2\,\hat{1}$. Right: $x = \hat{1} + \hat{2}$.

formulas listed in this appendix have been checked numerically by an exhaustive sum over all random walks of length $2n \leq 20$. In ref. [38], we present a proof of all the formulas of the next subsection, where the two endpoints are identical. The same method can be used to study the case where the endpoints are distinct, although we have not yet done so.

In the following, we give formulas for the four cases illustrated in the figures 8 and 9, which covers all the situations where 0 and x are separated by two hops at most. The simplest situation is when the point $x = 0$ and the path γ_{x0} is the null path (figure 8, left). In this case, we have[15]

$$\sum_{\gamma \in \mathbf{\Gamma}_{n_1,n_2}(0,0)} \big(\mathrm{Area}\,(\gamma)\big)^2 = \frac{(2(n_1+n_2))!}{n_1!^2 n_2!^2}\;\frac{n_1 n_2}{3}\,. \tag{C.1}$$

(We recall that $\mathbf{\Gamma}_{n_1,n_2}(0,x)$ is the set of all the paths that connect the point 0 to the point x and have n_1 hops in the direction $+x_1$ and n_2 hops in the direction $+x_2$.)

Next, let us consider the case where $x = \hat{x}_1$ and $\gamma_{x0} = \hat{x}_1^{-1}$ (i.e. a single hop in the $-x_1$ direction — see the figure 8, right). This case leads to

$$\sum_{\gamma \in \mathbf{\Gamma}_{n_1,n_2}(0,\hat{1})} \big(\mathrm{Area}\,(\gamma \otimes \hat{1}^{-1})\big)^2 = \frac{(2(n_1+n_2)-1)!}{n_1!(n_1-1)!n_2!^2}\;\frac{n_1 n_2}{3}\,. \tag{C.2}$$

For separations of two hops, there are two possibilities, shown in the figure 9. When the point x is located at $x = 2\,\hat{1}$ and the path connecting to 0 is made of two horizontal hops (figure 9, left), we have

$$\sum_{\gamma \in \mathbf{\Gamma}_{n_1,n_2}(0,2\,\hat{1})} \big(\mathrm{Area}\,(\gamma \otimes \hat{1}^{-2})\big)^2 = \frac{(2(n_1+n_2-1))!}{n_1!(n_1-2)!n_2!^2}\;\frac{(n_1+1) n_2}{3}\,. \tag{C.3}$$

When the point x is at $x = \hat{1}+\hat{2}$, and is connected to the point 0 by the path $\gamma_{x0} = \hat{2}^{-1}\hat{1}^{-1}$, we obtain

$$\sum_{\gamma \in \mathbf{\Gamma}_{n_1,n_2}(0,\hat{1}+\hat{2})} \big(\mathrm{Area}\,(\gamma \otimes \hat{2}^{-1}\hat{1}^{-1})\big)^2 = \frac{(2(n_1+n_2-1))!}{n_1!(n_1-1)!n_2!(n_2-1)!}\left(\frac{n_1 n_2}{3}+\frac{1}{6}\right). \tag{C.4}$$

C.2 Higher moments (for $x = 0$ and $\gamma_{x0} = 1$)

In the case of closed paths γ, it is also possible to obtain by empirical observation a simple formula for the moment of order 4 of the area,

$$\sum_{\gamma \in \mathbf{\Gamma}_{n_1,n_2}(0,0)} \big(\mathrm{Area}\,(\gamma)\big)^4 = \frac{(2(n_1+n_2))!}{n_1!^2 n_2!^2}\;\frac{n_1 n_2}{15}\big(7n_1 n_2 - (n_1+n_2)\big)\,, \tag{C.5}$$

[15] One can check that

$$\sum_{n_1=0}^{n} \frac{(2n)!}{n_1!^2(n-n_1)!^2}\;\frac{n_1(n-n_1)}{3} = \frac{(2n)!^2}{n!^4}\;\frac{n^2(n-1)}{6(2n-1)}\,,$$

which is nothing but the isotropic result for the variance of the areas in the set of all the closed random walks of lengths $2n$ (see [37], eqs. (1.4)–(1.5)).

that leads to the formula (1.6) of ref. [37] after summing over $0 \leq n_1 \leq n$ (with $n_2 = n - n_1$). Interestingly, the formulas for the moments are somewhat simpler if one considers only random paths with fixed numbers of hops in the $+x_1$ and $+x_2$ directions, rather that all the random paths that have a fixed length. Indeed, besides the expected combinatorial factor, these formulas seem to involve a polynomial in $n_{1,2}$, instead of a rational fraction in $n = n_1 + n_2$.

Similarly, we have obtained the following expressions for the moments of order 6, 8 and 10:

$$\begin{aligned}
\sum_{\gamma\in\boldsymbol{\Gamma}_{n_1,n_2}(0,0)} \big(\mathrm{Area}\,(\gamma)\big)^6 &= \frac{(2(n_1+n_2))!}{n_1!^2 n_2!^2}\,\frac{n_1n_2}{21}\big(31(n_1n_2)^2 \\
&\quad -15n_1n_2(n_1+n_2)+2(n_1+n_2)^2-(n_1+n_2)\big)\,, &\text{(C.6)}\\
\sum_{\gamma\in\boldsymbol{\Gamma}_{n_1,n_2}(0,0)} \big(\mathrm{Area}\,(\gamma)\big)^8 &= \frac{(2(n_1+n_2))!}{n_1!^2 n_2!^2}\,\frac{n_1n_2}{15}\big(127(n_1n_2)^3 \\
&\quad -134(n_1n_2)^2(n_1+n_2)+53n_1n_2(n_1+n_2)^2-6(n_1^3+n_2^3) \\
&\quad -40n_1n_2(n_1+n_2)+8(n_1+n_2)^2-3(n_1+n_2)\big)\,, &\text{(C.7)}\\
\sum_{\gamma\in\boldsymbol{\Gamma}_{n_1,n_2}(0,0)} \big(\mathrm{Area}\,(\gamma)\big)^{10} &= \frac{(2(n_1+n_2))!}{n_1!^2 n_2!^2}\,\frac{n_1n_2}{33}\big(2555(n_1n_2)^4 \\
&\quad -4778(n_1n_2)^3(n_1+n_2)+3745(n_1n_2)^2(n_1+n_2)^2 \\
&\quad -5290(n_1n_2)^2(n_1+n_2)-1282n_1n_2(n_1^3+n_2^3) \\
&\quad +1918n_1n_2(n_1+n_2)^2+120(n_1^2-n_2^2)^2 \\
&\quad -1403n_1n_2(n_1+n_2)-300(n_1^3+n_2^3) \\
&\quad +270(n_1+n_2)^2-85(n_1+n_2)\big)\,. &\text{(C.8)}
\end{aligned}$$

These observations suggest that we have in general

$$\sum_{\gamma\in\boldsymbol{\Gamma}_{n_1,n_2}(0,0)} \big(\mathrm{Area}\,(\gamma)\big)^{2k} = \frac{(2(n_1+n_2))!}{n_1!^2 n_2!^2}\,\mathcal{P}_{2k}(n_1,n_2)\,, \tag{C.9}$$

where $\mathcal{P}_{2k}(n_1, n_2)$ is a polynomial of degree $2k$, symmetric in (n_1, n_2), proportional to n_1n_2,with rational coefficients.[16] The leading term of this polynomial is of the form $c_k(n_1n_2)^{k-1}$, and the first five terms obtained above are consistent with (defining $c_0 \equiv 1$)

$$\sum_{k=0}^{\infty} c_k\,\frac{z^{2k}}{(2k)!} = \frac{z}{\sin(z)}\,, \tag{C.10}$$

in agreement[17] with the eq. (1.7) of ref. [37]. In a separate paper, ref. [38], we prove

[16] Such a polynomial has k^2 independent coefficients. If we evaluate the areas of all the closed paths with n_1, n_2 such that $n_1 + n_2 \leq n$, we get p^2 independent constraints if $n = 2p$ and $p(p + 1)$ independent constraints if $n = 2p + 1$. Therefore, in order to determine uniquely the polynomial $\mathcal{P}_{2k}$, it is sufficient to consider all the closed paths up to the length $2n = 4k$. This is why one can obtain all the moments up to $2k = 10$ by considering all the paths up to the length 20.

[17] In order to establish this connection, it is sufficient to notice that

$$\sum_{n_1=0}^{n} \frac{(2n)!}{n_1!^2(n-n_1)!^2}\,n_1^k = \frac{(2n)!^2}{n!^4}\left[\left(\frac{n}{2}\right)^k + \text{subleading terms in } n\right].$$

eq. (C.9) that gives the general structure of the moments. The approach used in this paper also provides an algorithm for calculating explicit the polynomial $\mathcal{P}_{2k}$ for small values of k, and we have checked the formulas (C.1), (C.5), (C.6), (C.7) and (C.8).

D Link with the almost-Mathieu operator

One can relate the statistics of closed random walks on $\mathbb{Z}^2$ to Euclidean lattice scalar QED on a 2-dimensional lattice by means of the discrete worldline formalism. In order to keep track of the areas enclosed by these random walks, one should add a magnetic field transverse to the plane in which the charged scalar particles live. Moreover, since we would like to distinguish the random walks according to the number of hops they make in the x and y directions, we need a rectangular lattice, with distinct lattice spacings a_1 and a_2 in the two directions. In order to write explicit expressions, we must choose a gauge for the magnetic field. A convenient choice is

$$A_1(i,j) = 0\,, \qquad A_2(i,j) = Bx = Ba_1 i\,, \tag{D.1}$$

where (i,j) are the integers that label the position of a point on the lattice. In the lattice formulation, this background electromagnetic field is implemented by gauge links[18]

$$U_1(i,j) = 1\,, \qquad U_2(i,j) = e^{iBa_2a_1 i}\,. \tag{D.2}$$

In the following, we denote $2\pi\beta \equiv Ba_1a_2$ the magnetic flux through an elementary plaquette of the lattice. With this gauge choice, the action of the inverse propagator on a test function reads

$$(G^{-1}\phi)_{i,j} = \frac{\phi_{i+1,j}+\phi_{i-1,j}-2\phi_{i,j}}{a_1^2} + \frac{e^{i2\pi\beta i}\phi_{i,j+1}+e^{-i2\pi\beta i}\phi_{i,j-1}-2\phi_{i,j}}{a_2^2}\,. \tag{D.3}$$

Moreover, the worldline representation of the propagator at equal points (the choice of the point 0 is arbitrary, and irrelevant since the magnetic field is homogeneous) reads

$$G(0,0) = -\frac{\mathbf{a}^2}{4}\sum_{n_1,n_2=0}^{\infty}\frac{h_1^{2n_1}h_2^{2n_2}}{4^{2(n_1+n_2)}}\sum_{\gamma\in\mathbf{\Gamma}_{n_1,n_2}(0,0)}\left[\prod_{\ell\in\gamma}U_\ell\right], \tag{D.4}$$

where we have defined

$$\frac{2}{\mathbf{a}^2} \equiv \frac{1}{a_1^2}+\frac{1}{a_2^2}\,, \qquad h_{1,2} \equiv \frac{\mathbf{a}^2}{a_{1,2}^2}\,. \tag{D.5}$$

(Note that $h_1+h_2=2$.) $\mathbf{\Gamma}_{n_1,n_2}(0,0)$ is the set of the closed paths (from $(0,0)$ to $(0,0)$) drawn on the lattice, that have exactly n_1 hops in the $+x$ direction and n_2 hops in the $+y$ direction (and therefore also $n_{1,2}$ hops in the $-x$ and $-y$ directions). The product of the link variables encountered along the closed path is also the exponential of the magnetic flux,

$$\prod_{\ell\in\gamma}U_\ell = e^{i2\pi\beta\,\mathrm{Area}\,(\gamma)}\,, \tag{D.6}$$

[18] For simplicity, the electrical charge is taken to be $e=1$.

where Area (γ) is the algebraic area enclosed by the path γ. Therefore, we have

$$G(0,0) = -\frac{\mathbf{a}^2}{4} \sum_{n_1,n_2=0}^{\infty} \frac{h_1^{2n_1} h_2^{2n_2}}{4^{2(n_1+n_2)}} \sum_{\gamma \in \mathbf{\Gamma}_{n_1,n_2}(0,0)} e^{i2\pi\beta \,\mathrm{Area}\,(\gamma)} \,, \tag{D.7}$$

and one can view the diagonal elements of the propagator in a magnetic field as a generating function for the distribution of the areas of closed loops on the lattice.

The links $\exp(i2\pi\beta i)$ depend only on the i coordinate. Therefore, one can perform a Fourier transform on the j coordinate. Let us define:

$$\phi_{i,k} \equiv \sum_{j=0}^{N-1} \phi_{i,j}\, e^{-i2\pi\frac{kj}{N}} \,, \tag{D.8}$$

where N is the number of lattice spacings in the j direction. The conjugate index k is also an integer in the range $[0, N-1]$. The inverse Fourier transform reads

$$\phi_{i,j} = \frac{1}{N} \sum_{k=0}^{N-1} \psi_{i,k}\, e^{i2\pi\frac{kj}{N}} \,, \tag{D.9}$$

If we consider an infinitely large lattice, $N \to +\infty$, one can use a continuous momentum variable $\nu \equiv 2\pi k/N$, so that the above equation becomes

$$\phi_{i,j} = \int_0^{2\pi} \frac{d\nu}{2\pi}\, \psi_{i,\nu}\, e^{i\nu j} \,. \tag{D.10}$$

By inserting this equation into eq. (D.3), we obtain

$$\Big[\big(\mathbf{a}^2 G^{-1} + 4\big)\psi\Big]_{i,\nu} = \sum_{i'} \int \frac{d\nu'}{2\pi}\, H^{(\beta)}_{i\nu,i'\nu'}\, \psi_{i',\nu'} \,, \tag{D.11}$$

where $H^{(\beta)}$ is known as the (anisotropic) almost-Mathieu operator:

$$\Big(H^{(\beta)}\Big)_{i\nu,i'\nu'} \equiv \Big[h_1(\delta_{i,i'+1} + \delta_{i,i'-1}) + 2\,h_2 \cos(2\pi\beta i + \nu)\,\delta_{i,i'}\Big]\, 2\pi\delta(\nu - \nu') \,. \tag{D.12}$$

This Hamiltonian has been the subject of many studies, both for its practical interest in models of the quantum Hall effect [40–43], and for its intrinsic mathematical interest [44–51] as an example of quasi-periodic Hamiltonian (when β is an irrational number).

From eq. (D.11), we get that

$$G = \mathbf{a}^2 \big[H^{(\beta)} - 4\big]^{-1} = -\frac{\mathbf{a}^2}{4} \sum_{n=0}^{\infty} \left(\frac{H^{(\beta)}}{4}\right)^n \,. \tag{D.13}$$

Since the external magnetic field is homogeneous, the diagonal elements of the propagator can be obtained by taking the trace (divided by the number of lattice sites in the i direction). Thus, by comparing eqs. (D.7) and (D.13) we recover the well-known connection [52–54] between the trace of $\big(H^{(\beta)}\big)^n$ and the distribution of the areas of the closed random walks of length n,

$$\sum_{n_1+n_2=n} h_1^{2n_1} h_2^{2n_2} \sum_{\gamma \in \mathbf{\Gamma}_{n_1,n_2}(0,0)} e^{i2\pi\beta\,\mathrm{Area}\,(\gamma)} = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \int_0^{2\pi} \frac{d\nu}{2\pi} \Big[\big(H^{(\beta)}\big)^{2n}\Big]_{i\nu,i\nu} \,. \tag{D.14}$$

E $\langle\phi_a^*(0)\phi_a(0)\rangle$ from lattice perturbation theory

In this appendix, we calculate $\langle\phi_a^*(x)\phi_a(x)\rangle$ in lattice perturbation theory in order to make the connection with our results obtained in the lattice worldline representation. In lattice perturbation theory (see ref. [18] for a comprehensive review), the expectation value of any operator $\mathcal{O}$ can be calculated perturbatively according to

$$\langle\mathcal{O}\rangle = \frac{\left[\int D\phi D\phi^\dagger\right]\,\mathcal{O}[\phi,\phi^\dagger]\,e^{-S[\phi^\dagger,\phi]}}{\int\left[D\phi D\phi^\dagger\right]\,e^{-S[\phi^\dagger,\phi]}}\,. \tag{E.1}$$

In our case, the action reads

$$S[\phi^\dagger,\phi] \equiv S_0[\phi^\dagger,\phi] + S_{\rm int}[\phi^\dagger,\phi,U]\,, \tag{E.2}$$

with

$$S_0[\phi^\dagger,\phi] = -V_a\sum_x \phi_x^\dagger\sum_{r=1}^{d}\frac{1}{a_r^2}\left[\phi(x+a_r\hat{r})+\phi(x-a_r\hat{r})-2\phi(x)\right], \tag{E.3}$$

$$S_{\rm int}[\phi^\dagger,\phi] = -V_a\sum_x\sum_{r=1}^{d}\frac{1}{a_r^2}\Big[\phi^\dagger(x)(U_r(x)-\mathbb{I})\phi(x+a_r\hat{r})+\phi^\dagger(x+a_r\hat{r})(U_r^{-1}(x)-\mathbb{I})\phi(x)\Big], \tag{E.4}$$

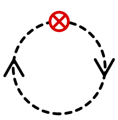
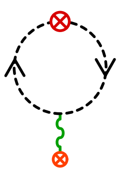
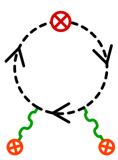
and $V_a \equiv \prod_{r=0}^{d} a_i$. In the limit of an infinitely large lattice, the free propagator reads

$$\begin{aligned} G_0(x,y) &= \left[\prod_{r=1}^{d}\int_{-\frac{\pi}{a_r}}^{\frac{\pi}{a_r}}\frac{dp_r}{2\pi}e^{ip_r(x_r-y_r)}\right]\frac{1}{2\sum_{l=1}^{d}\frac{1}{a_l^2}[1-\cos(p_l a_l)]} \\ &= \frac{1}{V_a}\left[\prod_{r=1}^{d}\int_{-\pi}^{\pi}\frac{dp_r}{2\pi}e^{ip_r(m_r-n_r)}\right]\underbrace{\frac{1}{2\sum_{l=1}^{d}\frac{1}{a_l^2}(1-\cos p_l)}}_{\widehat{G}(\{p_l\})}\,, \end{aligned} \tag{E.5}$$

with $x_r = m_r a_r, y_r = n_r a_r$. Since it is a Green's function of the discrete Euclidean d'Alembertian operator, it satisfies the identity

$$-\sum_{r=1}^{d}\frac{1}{a_r^2}\left[G_0(x+a_r\hat{r},y)+G_0(x-a_r\hat{r},y)-2G_0(x,y)\right] = \frac{1}{V_a}\delta_{m_r,n_r}\,. \tag{E.6}$$

Let us now calculate the first two terms of the expansion of $\langle\phi_a^*(0)\phi_a(0)\rangle$ in powers of the external field,

$\langle\phi_a^*(0)\phi_a(0)\rangle =$  $+$  $+$  $+\cdots\equiv I_0+I_1+I_2+\cdots\,.$ (E.7)

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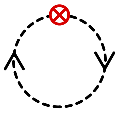
The term of order zero is given by:

$$I_0 \equiv$$  $$= \frac{\mathrm{tr_{adj}}(1)}{2dV_a} \int_{-\pi}^{\pi} \frac{d^dp}{(2\pi)^d} \frac{1}{\mathbf{a}^{-2} - \frac{1}{d}\sum\limits_{l=1}^{d} a_l^{-2}\cos p_l}$$

$$= \frac{\mathbf{a}^2\ \mathrm{tr_{adj}}(1)}{2dV_a} \int_0^{\infty} dt\ e^{-t} \int_{-\pi}^{\pi} \frac{d^dp}{(2\pi)^d}\ \exp\left(\sum_{l=1}^{d} \frac{h_r t}{d} \cos p_l\right)$$

$$= \frac{\mathbf{a}^2\ \mathrm{tr_{adj}}(1)}{2dV_a} \int_0^{\infty} dt\ e^{-t} \prod_{r=1}^{d} I_0\left(\frac{h_r t}{d}\right), \tag{E.8}$$

which is equivalent to eqs. (4.6) and (4.11).

Next, we calculate the corrections to $\langle \phi_a^*(0)\phi_a(0)\rangle$ of order $\mathcal{O}(g^2a^2)$. By using the power series expansion of the link variable,

$$U_r(x) = e^{-iga_rA_r(x)} = \mathbb{I} - iga_rA_r(x) - \frac{g^2}{2}a_r^2A_r^2(x) + \cdots, \tag{E.9}$$

and keeping only terms of $\mathcal{O}(g^2)$, we find

$$I_1 + I_2 = \frac{g^2}{2}\int \frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d} \sum_{r,s=1}^{d} \widetilde{A}_r^a(k_1)\widetilde{A}_s^a(k_2) I_{rs}(k_1,k_2) + \mathcal{O}(g^3), \tag{E.10}$$

with

$$\begin{aligned} I_{rs}(k_1,k_2) = \frac{1}{V_a}\int_{-\pi}^{\pi}\frac{d^dp}{(2\pi)^d}\ \widehat{G}(\{p_l\})\Bigg[&\frac{1}{a_ra_s}\widehat{G}(\{p_l+k_{1l}a_l\})\widehat{G}(\{p_l-k_{2l}a_l\}) \\ &\times\left(e^{i(p_r-p_s)}-e^{ip_r+ip_s-ik_{2s}a_s}+e^{-i(p_r+k_{1r}a_r)+i(p_s-k_{2s}a_s)}-e^{-i(p_r+k_{1r}a_r)-ip_s}\right) \\ &-\frac{\delta_{rs}}{2}\widehat{G}(\{p_l+(k_{1l}+k_{2l})a_l\})(e^{ip_r}+e^{-ip_r-i(k_{1r}+k_{2r})a_r})\Bigg]. \end{aligned} \tag{E.11}$$

The terms of order g^2a^2 can be obtained by performing a series expansion of I_{rs} in terms of $k_{1,2}a$. It is easy to show that

$$I_{rs}(k_1,k_2) = I_{1rs} + I_{2,1rs} + I_{2,2rs} + I_{2,3rs} + I_{2,4rs} + \mathcal{O}(a^3), \tag{E.12}$$

with

$$I_{1rs} = -\frac{\delta_{rs}}{2V_a}\sum_{i=1}^{d}\int_{-\pi}^{\pi}\frac{d^dp}{(2\pi)^d}\ (k_{1i}+k_{2i})^2a_i^2\ \widehat{G}(\{p_l\})\frac{\partial^2\widehat{G}(\{p_l\})}{\partial p_i^2}\ \cos p_r,$$

$$I_{2,1rs} = \frac{1}{V_a}\sum_{i,j=1}^{d}\int_{-\pi}^{\pi}\frac{d^dp}{(2\pi)^d}\ \frac{2a_ia_j}{a_ra_s}\ (k_{1i}k_{1j}+k_{2i}k_{2j})\ \widehat{G}^2(\{p_l\})\times\frac{\partial^2\widehat{G}(\{p_l\})}{\partial p_i\partial p_j}\sin p_r\sin p_s,$$

$$I_{2,2rs} = -\frac{4}{V_a}\sum_{i,j=1}^{d}\int_{-\pi}^{\pi}\frac{d^dp}{(2\pi)^d}\ \frac{a_ia_j}{a_ra_s}\ k_{1i}k_{2j}\widehat{G}(\{p_l\})\times\frac{\partial\widehat{G}(\{p_l\})}{\partial p_i}\frac{\partial\widehat{G}(\{p_l\})}{\partial p_j}\ \sin p_r\sin p_s,$$

$$
\begin{aligned}
I_{2,3rs} &= \frac{2}{V_a} \int_{-\pi}^{\pi} \frac{d^dp}{(2\pi)^d}\, \widehat{G}^2(\{p_l\}) \left[(k_{1s}-k_{2s})k_{1r}\, \frac{\partial \widehat{G}(\{p_l\})}{\partial p_s} \sin p_s \cos p_r \right. \\
&\qquad \left. -(k_{1r}-k_{2r})k_{2s}\, \frac{\partial \widehat{G}(\{p_l\})}{\partial p_r} \cos p_s \sin p_r \right], \\
I_{2,4rs} &= -\frac{1}{V_a} \int_{-\pi}^{\pi} \frac{d^dp}{(2\pi)^d}\, \widehat{G}^3(\{p_l\}) \left[\delta_{rs}(k_{1r}^2+k_{2r}^2)\sin^2 p_r \right. \\
&\qquad \left. +k_{1r}k_{2s}\left(\cos p_r \cos p_s + \sin^2 p_r \delta_{sr}\right) \right].
\end{aligned} \tag{E.13}
$$

In order to simplify the above integrals, we need the following identities

$$
\begin{aligned}
\frac{\partial \widehat{G}(\{p_l\})}{\partial p_r} &= -\frac{2}{a_r^2} \sin p_r\, \widehat{G}^2(\{p_l\}), \\
\frac{\partial^2 \widehat{G}(\{p_l\})}{\partial p_r \partial p_s} &= \frac{8}{a_r^2 a_s^2} \sin p_r \sin p_s\, \widehat{G}^3(\{p_l\}) - \frac{2}{a_r^2} \cos p_r\, \widehat{G}^2(\{p_l\})\delta_{rs}, \\
&= 2\widehat{G}^{-1}(\{p_l\}) \frac{\partial \widehat{G}(\{p_l\})}{\partial p_r} \frac{\partial \widehat{G}(\{p_l\})}{\partial p_s} - \frac{2}{a_r^2} \cos p_r\, \widehat{G}^2(\{p_l\})\delta_{rs}.
\end{aligned} \tag{E.14}
$$

Moreover, the terms that are total derivatives with respect to any component of p in the above integrands always give vanishing contributions. After some algebra we find

$$
\begin{aligned}
I_{1rs} &= \frac{\delta_{rs}}{3V_a} \int_{-\pi}^{\pi} \frac{d^dp}{(2\pi)^d}\, \widehat{G}^3(\{p_l\}) \left[\sum_{i=1}^{d} (k_{1i}+k_{2i})^2 \cos p_i \cos p_r \right. \\
&\qquad \left. +2(k_{1r}+k_{2r})^2 \sin^2 p_r \right], \\
I_{2,1rs} &= \frac{1}{3V_a} \int_{-\pi}^{\pi} \frac{d^dp}{(2\pi)^d}\, \widehat{G}^3(\{p_l\}) \left[2(k_{1r}k_{1s}+k_{2r}k_{2s}) \cos p_r \cos p_s \right. \\
&\qquad \left. -(k_{1r}^2+k_{2r}^2)\delta_{rs} \sin^2 p_r - \sum_{i=1}^{d}(k_{1i}^2+k_{2i}^2) \cos p_i \cos p_r \delta_{rs} \right], \\
I_{2,2rs} &= -\frac{1}{3V_a} \int_{-\pi}^{\pi} \frac{d^dp}{(2\pi)^d}\, \widehat{G}^3(\{p_l\}) \left[(k_{1r}k_{2s}+k_{1s}k_{2r}) \cos p_r \cos p_s \right. \\
&\qquad \left. -3k_{1r}k_{2r}\delta_{rs} \sin^2 p_r + \sum_{i=1}^{d} k_{1i}k_{2i} \cos p_i \cos p_r \delta_{rs} \right], \\
I_{2,3rs} &= -\frac{2}{3V_a} \int_{-\pi}^{\pi} \frac{d^dp}{(2\pi)^d}\, \widehat{G}^3(\{p_l\}) \left[(k_{1s}k_{1r}+k_{1s}k_{1r}-2k_{1r}k_{2s}) \cos p_s \cos p_r \right. \\
&\qquad \left. -(k_{1r}^2+k_{2r}^2-2k_{1r}k_{2r})\delta_{rs} \sin^2 p_r \right].
\end{aligned} \tag{E.15}
$$

As a result, up to $\mathcal{O}(g^2a^2)$, we have

$$
I_{rs}(k_1,k_2) = \frac{1}{3V_a} \int_{-\pi}^{\pi} \frac{d^dp}{(2\pi)^d}\, \widehat{G}^3(\{p_l\}) \left[\delta_{rs} \sum_{i=1}^{d} k_{1i}k_{2i} \cos p_i \cos p_r - k_{1s}k_{2r} \cos p_r \cos p_s \right], \tag{E.16}
$$

and

$$
\begin{aligned}
I_1 + I_2 &= -\frac{g^2}{12V_a}\sum_{r,s}[\partial_r A_s^a(x) - \partial_s A_r^a(x)]^2 \int_{-\pi}^{\pi} \frac{d^d p}{(2\pi)^d}\, \widehat{G}^3(\{p_l\}) \cos p_r \cos p_s \\
&= -\frac{g^2 \mathbf{a}^6}{192 d^3 V_a}\sum_{r\neq s}[\partial_r A_s^a(x) - \partial_s A_r^a(x)]^2 \\
&\qquad \times \int_0^\infty dt\, t^2\, e^{-t}\, I_1\!\left(\frac{h_r t}{d}\right) I_1\!\left(\frac{h_s t}{d}\right) \prod_{i\neq r,s} I_0\!\left(\frac{h_i t}{d}\right),
\end{aligned}
$$

where we have used the integral

$$
\frac{1}{x^3} = \frac{1}{2}\int_0^\infty dt\, t^2\, e^{-tx}\,. \tag{E.17}
$$

In summary, we have

$$
\begin{aligned}
\langle \phi_a^*(0)\phi_a(0)\rangle &= \frac{\mathbf{a}^2\, \mathrm{tr}_{\mathrm{adj}}(1)}{2dV_a}\int_0^\infty dt\, e^{-t} \prod_{r=1}^{d} I_0\!\left(\frac{h_r t}{d}\right) \\
&\quad -\frac{g^2 \mathbf{a}^6}{192 d^3 V_a}\sum_{r\neq s}[\partial_r A_s^a(x) - \partial_s A_r^a(x)]^2 \\
&\qquad \times \int_0^\infty dt\, t^2\, e^{-t}\, I_1\!\left(\frac{h_r t}{d}\right) I_1\!\left(\frac{h_s t}{d}\right) \prod_{i\neq r,s} I_0\!\left(\frac{h_i t}{d}\right) + \mathcal{O}(g^2 a^3)\,.
\end{aligned} \tag{E.18}
$$

This result is identical to the formulas obtained in the worldline formalism for this expectation value (see the eqs. (4.6) and (4.20)). Therefore, one can view this alternate derivation as an indirect analytical proof of the formula (C.1), that we have used in obtaining the next-to-leading order coefficient in the anisotropic case. In principle, one could apply the same technique in order to prove all the other formulas conjectured in the appendix C, although this appears to be a rather intricate task.

F Continuous time and discrete space

The extreme case of anisotropy is realized when one of the coordinates (time in our discussion) is treated as a continuous variable, while the others remain discretized on a lattice of spacing a. In order to simplify the treatment of the background field, it is useful to assume that the temporal gauge is used, $A_0 = 0$. Therefore, the inverse propagator is

$$
D^2 \equiv g_{00}\partial_0^2 - \sum_{i=1}^{d_s} \nabla_i^+ \nabla_i^-\,, \tag{F.1}
$$

where $\nabla_i^\pm$ are the forward and backward discrete covariant derivatives on a grid of lattice spacing a, and $d_s \equiv d-1$ the number of spatial dimensions. At this point, we have also kept undetermined the 00 component of the metric tensor, g_{00}, in order to discuss later the difference in this formalism between the Minkowski and the Euclidean metric.

For the purpose of this discussion, we can first ignore the background field completely, and reintroduce it later via the Wilson loop made of the gauge links accumulated along the random walk. Let us recall that

$$\nabla_i^+\nabla_i^- f(x) = \frac{f(x+\hat{\imath})+f(x-\hat{\imath})}{a^2} - \frac{2\,f(x)}{a^2}\,. \tag{F.2}$$

It is convenient to write the inverse propagator as follows

$$D^2 = \frac{2d_s}{a^2}\left(\frac{g_{00}a^2}{2d_s}\partial_0^2+1\right)\left[1-\left(\frac{g_{00}a^2}{2d_s}\partial_0^2+1\right)^{-1}\boldsymbol{H}\right], \tag{F.3}$$

where we denote

$$\boldsymbol{H} f(x) \equiv \sum_{i=1}^{d_s} \frac{f(x+\hat{\imath})+f(x-\hat{\imath})}{2d_s} \tag{F.4}$$

the operator $\boldsymbol{H}$ generates the hops for a random walk on a square lattice in d_s dimensions. Using eq. (F.3), we can write the inverse propagator as

$$\begin{aligned}\frac{2d_s}{a^2}\frac{1}{D^2} = &\left(\frac{g_{00}a^2}{2d_s}\partial_0^2+1\right)^{-1} + \left(\frac{g_{00}a^2}{2d_s}\partial_0^2+1\right)^{-1}\boldsymbol{H}\left(\frac{g_{00}a^2}{2d_s}\partial_0^2+1\right)^{-1}\\ &+\left(\frac{g_{00}a^2}{2d_s}\partial_0^2+1\right)^{-1}\boldsymbol{H}\left(\frac{g_{00}a^2}{2d_s}\partial_0^2+1\right)^{-1}\boldsymbol{H}\left(\frac{g_{00}a^2}{2d_s}\partial_0^2+1\right)^{-1}+\cdots\end{aligned} \tag{F.5}$$

In this form, the inverse propagator appears as a sum of terms, each of which is an alternating product of $\boldsymbol{H}$ (i.e. single hops on the lattice that represents space) and of the inverse of the operator $\frac{g_{00}a^2}{2d_s}\partial_0^2+1$. We shall now rewrite this object in a way that clarifies its physical meaning. Let us start with its heat kernel representation

$$\frac{1}{\frac{g_{00}a^2}{2d_s}\partial_0^2+1} = \int_0^\infty dt\ \exp\left(-t\left(1+\frac{g_{00}a^2}{2d_s}\partial_0^2\right)\right). \tag{F.6}$$

The second step is the following identity

$$e^{\frac{\alpha}{2}\partial^2} = \int_{-\infty}^{+\infty}\frac{dz}{\sqrt{2\pi\alpha}}\ e^{-\frac{z^2}{2\alpha}}\ e^{z\partial}\,, \tag{F.7}$$

or more explicitly

$$e^{\frac{\alpha}{2}\partial^2} f(x) = \int_{-\infty}^{+\infty}\frac{dz}{\sqrt{2\pi\alpha}}\ e^{-\frac{z^2}{2\alpha}}\ f(x+z)\,. \tag{F.8}$$

In words, the operator $\exp(\frac{\alpha}{2}\partial^2)$ is a diffusion operator that smears the target function by convolution with a Gaussian of variance α. But this interpretation is only possible if α is a positive real number. The closest formula if α is negative would be

$$e^{\frac{\alpha}{2}\partial^2} f(x) \underset{\alpha<0}{=} \int_{-\infty}^{+\infty}\frac{dz}{\sqrt{2\pi|\alpha|}}\ e^{-\frac{z^2}{2|\alpha|}}\ f(x+iz)\,, \tag{F.9}$$

but it requires that we complexify the variable x. This is precisely the situation we face when employing this formula to rewrite the inverse propagator: g_{00} must be negative in order to obtain a properly normalized Gaussian. If we start from the Minkowski metric ($g_{00} = +1$), we can still get a Gaussian with the "correct sign" if we complexify the time. The conclusion of this digression is that the diffusive interpretation of the time evolution of a quantum system is only possible with imaginary time.

From now on, we assume that $g_{00} = -1$. Eq. (F.6) can be rewritten as

$$\frac{1}{1-\frac{a^2}{2d_s}\partial_0^2} f(x_0) = \int_0^\infty dt \, \sqrt{\frac{d_s}{2\pi t a^2}} \int_{-\infty}^{+\infty} dz \; e^{-t} \; e^{-\frac{d_s z^2}{2a^2 t}} \; f(x_0+z) \,. \tag{F.10}$$

In the right hand side, we recognize an integral over diffusion processes of "duration" t, weighted by a factor $\exp(-t/2)$ that suppresses the contribution of diffusions longer than $t \sim 1$. Under such a diffusion, the time x_0 can shift by an amount of order a (the spatial lattice spacing).

The interpretation of eq. (F.5) is now quite transparent. The inverse of D^2 is obtained by intertwining continuous diffusions in imaginary time (of arbitrary lengths, but weighted in such a way that the displacement in time is of order a) and single hops in one of the spatial directions. Note that in eq. (F.10), the integration over the length t can be done analytically by using

$$\int_0^\infty \frac{dt}{\sqrt{t}} \; e^{-t} \; e^{-c/t} = \sqrt{\pi} \; e^{-2\sqrt{c}} \,. \tag{F.11}$$

Therefore, we can write

$$\frac{1}{1-\frac{a^2}{2d_s}\partial_0^2} f(x_0) = \frac{\gamma}{2} \int_{-\infty}^{+\infty} dz \; e^{-\gamma|z|} \; f(x_0+z) \,, \tag{F.12}$$

where we denote $\gamma \equiv \sqrt{2d_s/a^2}$. We see that the application of this operator is a convolution with a Laplace distribution. The typical width of the resulting smearing is $z \sim \gamma^{-1} \sim a$. In other words, the *spatial* lattice spacing also controls the typical size of the jumps in time that are interspersed between the jumps on the spatial lattice. This is consistent with our conclusion of the section 4, that the results become independent of the smallest lattice spacing when it becomes infinitesimally small (the continuous-time description adopted in this appendix corresponds to a discrete description of time with a time interval that goes to zero). In particular, the ultraviolet behavior in this limit is controlled by the next-to-smallest lattice spacing.

In the series expansion of the inverse propagator, we need to evaluate the $n+1$-th power of this operator,[19] acting on a starting distribution of the form $\delta(x_0)$,

$$\begin{aligned} \alpha_n(x_0) &\equiv \left[\frac{1}{1-\frac{a^2}{2d_s}\partial_0^2}\right]^{n+1} \delta(x_0) \\ &= \left(\frac{\gamma}{2}\right)^{n+1} \int dz_1 \cdots dz_{n+1} \; e^{-\gamma(|z_1|+\cdots+|z_{n+1}|)} \; \delta(x_0+z_1+\cdots+z_{n+1}) \end{aligned}$$

[19]The choice of the $A_0 = 0$ gauge is crucial here. Indeed, if we had a non-zero position dependent A_0 background field, the hops in the time direction would not commute with the spatial hops, and it would be impossible to collect them as the $n+1$-th power of a single temporal hop operator.

$$
\begin{aligned}
&= \left(\frac{\gamma}{2}\right)^{n+1} \int_{-\infty}^{+\infty} \frac{du}{2\pi}\, e^{iux_0} \left[\int dz\, e^{-\gamma|z|+iuz}\right]^{n+1} \\
&= \left(\frac{\gamma}{2}\right)^{n+1} \int_{-\infty}^{+\infty} \frac{du}{2\pi}\, e^{iux_0} \left[\frac{2\gamma}{u^2+\gamma^2}\right]^{n+1} .
\end{aligned} \tag{F.13}
$$

For $x_0 = 0$, the evaluation of the integral is straightforward and one obtains

$$
\alpha_n(0) = \frac{\gamma}{2}\, \frac{(2n)!}{4^n\, n!^2} . \tag{F.14}
$$

The second factor is the probability that a random walk in one dimension returns to the origin after $2n$ steps.

Using these results, we obtain the following expression for the leading term of $\langle \phi_a^*(0)\phi_a(0)\rangle$,

$$
\langle \phi^*(0)\phi(0)\rangle = \frac{1}{2d_s a}\, \widetilde{\boldsymbol{C}}_0\, \mathrm{tr}_{\mathrm{adj}}\,(1) + \cdots , \tag{F.15}
$$

with

$$
\widetilde{\boldsymbol{C}}_0 \equiv \frac{\gamma}{2} \sum_{n=0}^{\infty} \frac{(4n)!}{4^{2n}(2n)!}\, \frac{1}{(2d_s)^{2n}} \sum_{n_1+\cdots+n_{d_s}=n} \frac{(2n)!}{n_1!^2 \cdots n_{d_s}!^2} . \tag{F.16}
$$

In this summation, we must have an even number $2n$ of spatial hops since their combination must form a closed loop on the spatial lattice. The first factor is therefore eq. (F.14) evaluated for $2n$ instead of n. The second factor counts the number of closed random walks of length $2n$ in d_s dimensions. In order to disentangle the sums over $n_1, \cdots, n_{d_s}$, we need a modified form of the Borel transformation, tuned to cancel the n-dependent prefactor. Notice first that

$$
\frac{1}{\sqrt{\pi}} \int_0^{\infty} \frac{d\tau}{\sqrt{\tau}}\, e^{-\tau}\, \tau^{2n} = \frac{(4n)!}{4^{2n}(2n)!} . \tag{F.17}
$$

This leads immediately to

$$
\widetilde{\boldsymbol{C}}_0 = \frac{\gamma}{2} \frac{1}{\sqrt{\pi}} \int_0^{\infty} \frac{d\tau}{\sqrt{\tau}}\, e^{-\tau}\, I_0^{d_s}\!\left(\frac{\tau}{d_s}\right) , \tag{F.18}
$$

and finally to

$$
\langle \phi^*(0)\phi(0)\rangle \underset{a_d \ll a_{1\cdots d-1} \to 0}{=} \frac{\mathrm{tr}_{\mathrm{adj}}\,(1)}{2a^{d-2}}\, \frac{1}{\sqrt{2\pi d_s}} \int_0^{\infty} \frac{d\tau}{\sqrt{\tau}}\, e^{-\tau}\, I_0^{d_s}\!\left(\frac{\tau}{d_s}\right) , \tag{F.19}
$$

which is identical to the formula of the footnote 13, that was obtained by taking the limit $a_d \to 0$ in a discrete time formulation.

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